Factoring

Factoring is important since

- it is used to solve some quadratic equations of the form $x^2 + bx + c = 0$ (we will learn another technique involving the quadratic formula later),
- it is used to simplify rational expressions,
- as the complement to the distributive property, it is one of the most important algebraic techniques to effectively work with algebraic expressions,
- it can be used to factor higher degree polynomials and work with more complicated expressions like $8x^3y + 16x^2y^2 - 24x^3y^3$.

Common factors in terms

Why do this? Usually you want the greatest common factor so you can work with smaller numbers:

$$8x^3y + 16x^2y^2 - 24x^3y^3 = 8x^2y(x + 2y - 3xy^2).$$

Factoring by Grouping

This is sort of like using the distribution property in the other direction:

$$(3y - 8)(2x - 7) = 3y(2x - 7) - 8(2x - 7) \quad \text{(distribution property)}$$
$$3y(2x - 7) - 8(2x - 7) = (3y - 8)(2x - 7) \quad \text{(factoring by grouping)}$$

Factoring trinomials of form $x^2 + bx + c$

$x^2 + bx + c = (x + m)(x + n)$ where m and n are two numbers whose product is c and sum is b.

Example Factor $x^2 + 5x - 6$.

Look for two numbers whose product is -6 and sum is 5: the numbers are $+6, -1$:

$$c^2 + 5x - 6 = (x+ \qquad)(x+ \qquad) = (x + 6)(x + (-1)) = (x + 6)(x - 1)$$

Always multiply out to determine if you have the correct answer:

$$(x + 6)(x - 1) = x(x - 1) + 6(x - 1) = x^2 - x + 6x - 6 = x^2 + 5x - 6.$$

Factoring trinomials of form $ax^2 + bx + c$

This is more involved, and requires the use of the *trial and error method*, or *the grouping method*. I prefer the grouping method, but both will work. I don't care which one you use.

The Grouping Method to factor trinomials of form $ax^2 + bx + c$:

1. Determine the grouping number ac.
2. Find two numbers whose product is ac and sum is b.
3. Use these numbers to write bx as the sum of two terms.
4. Factor by grouping.
5. Check your answer by multiplying out.

Example Factor $2x^2 - 7x + 6$.

Factor by grouping. Find two numbers whose sum is -7 and product is $2 \times 6 = 12$: $-3, -4$.

$$\begin{aligned} 2x^2 - 7x + 6 &= 2x^2 - 3x - 4x + 6 \text{ rewrite middle term using your two numbers} \\ &= x(2x-3) - 2(2x-3) \text{ common factors: the first two terms together and the last two together} \\ &= (x-2)(2x-3) \text{ factor by grouping} \end{aligned}$$

Multiply out to check:

$$(x-2)(2x-3) = x(2x-3) - 2(2x-3) = 2x^2 - 3x - 4x + 6 = 2x^2 - 7x + 6$$

How to "Find two numbers whose product is c and sum is b"

Trial and error–if the numbers are small enough, you can usually guess your two numbers.

A slightly more systematic approach is to list the factors of the product until you either find your two numbers or determine there aren't any (the trinomial is then prime).

For example, if we want two numbers whose product is -154 and whose sum is 15 we can do the following:

Product	Factor	Sum
-154	$(77)(-2)$	75
-154	$(14)(-11)$	3
-154	$(7)(-22)$	-15
-154	$(-7)(22)$	15

Stop once you have found the two numbers, in this case -7 and 22.

Obviously, for large numbers this can be a lot of work. This is one reason why it is suggested that you factor common factors at the beginning, so the numbers you are working with are smaller and the "Find two numbers whose..." step is therefore easier.

Advice

Factoring might take multiple steps. Focus on making each step mathematically correct instead of finding the fastest way to the factorization. There are multiple ways to do each problem, each of which can be correct.

For example, consider the following factorization:

$$\begin{aligned} 5x^2 - 3xy - 10x + 6y &= x(5x-3y) + 2(-5x+3y) && (1) \\ &= x(5x-3y) - 2(5x-3y) && (2) \\ &= (x-2)(5x-3y) && (3) \end{aligned}$$

Some texts would say that step 1 is wrong because you should have factored -2 out of the last two terms. How are you supposed to know you should factor -2 instead of 2 at that point? You only know this once you get to step 2 and see that if you factor a -1 out of the second term you will be able to factor $5x - 3y$ out of each term. The solution presented above is mathematically correct at each step and gets to the right answer–there is nothing "wrong" about it!

Example Find a polynomial that describes how much greater the perimeter of the square is than the circumference of the circle for a circle that is inscribed in a square with sides of length x. Inscribed means the sides of the circle touch all four sides of the square.

The perimeter of the square is $x + x + x + x = 4x$.

The circumference of the circle is $2\pi(\text{radius}) = 2\pi\left(\frac{x}{2}\right) = \pi x$.

The difference is $4x - \pi x = (4 - \pi)x$.

Example Factor $3x^2 - 33x + 54$.

Factor a 3 first to get $a = 1$. Two numbers whose product is 18 and sum is -11: -2, -9.

$$\begin{aligned} 3x^2 - 33x + 54 &= 3(x^2 - 11x + 18) \\ &= 3(x-2)(x-9) \end{aligned}$$

Alternate solution: If you didn't notice you could factor a 3 out of each term, you can factor by grouping.

Find two numbers whose product is $3 \times 54 = 162$ and whose sum is -33: -27 and -6.

Now write the $-33x$ term as two terms based on the numbers you found.

$$\begin{aligned} 3x^2 - 33x + 54 &= 3x^2 - 27x - 6x + 54 \\ &\quad \text{(red terms have a factor of } 3x\text{)} \\ &\quad \text{(blue terms have a factor of } -6\text{)} \\ &= 3x(x-9) - 6(x-9) \\ &\quad \text{(both terms have a factor of } x-9\text{)} \\ &= (3x-6)(x-9) \end{aligned}$$

Check: $(3x-6)(x-9) = 3x^2 - 6x - 27x + 54 = 3x^2 - 33x + 54$.

Special Cases of Factoring

Difference of Two Squares

$$a^2 - b^2 = (a-b)(a+b).$$

Perfect Square (two cases)

$$\begin{aligned} a^2 + 2ab + b^2 &= (a+b)^2, \\ a^2 - 2ab + b^2 &= (a-b)^2. \end{aligned}$$

Sum and Difference of Cubes

$$\begin{aligned} a^3 + b^3 &= (a+b)(a^2 - ab + b^2), \\ a^3 - b^3 &= (a-b)(a^2 + ab + b^2). \end{aligned}$$

These should be used as formulas you have memorized, not something you "work out". You can "work out" the difference of square and perfect square cases using the grouping method of factoring if you have to, but the sum and difference of cubes must be used as memorized formulas.

Example Factor $9x^2 - 42xy + 49y^2$.

Notice that the first term and last term look like squares:
$9x^2 = (3x)^2 = a^2$ (so $a = 3x$) and $49y^2 = (7y)^2 = b^2$ (so $b = 7y$).
Check if the middle term is $2ab = 2(3x)(7y) = 42xy$. Yes! So this is a perfect square (difference).

$$\begin{aligned} 9x^2 - 42xy + 49y^2 &= a^2 - 2ab + b^2 \quad \text{identify as a perfect square (difference)} \\ &= (a-b)^2 \quad \text{perfect square (difference) formula} \\ &= (3x - 7y)^2 \quad \text{sub back in } a = 3x \text{ and } b = 7y. \end{aligned}$$

Example Factor $\dfrac{1}{256} + \dfrac{11}{40}x + \dfrac{121}{25}x^2$.

Notice that the first term and last term look like squares:
$\dfrac{1}{256} = \left(\dfrac{1}{16}\right)^2 = a^2$ (so $a = \frac{1}{16}$) and $\dfrac{121}{25}x^2 = \left(\dfrac{11}{5}x\right)^2 = b^2$ (so $b = \frac{11}{5}x$).
Check if the middle term is $2ab = 2\left(\dfrac{1}{16}\right)\left(\dfrac{11}{5}x\right) = \dfrac{11}{40}x$. Yes! So this is a perfect square (sum).

$$\begin{aligned}
\frac{1}{256} + \frac{11}{40}x + \frac{121}{25}x^2 &= a^2 + 2ab + b^2 \quad \text{identify as a perfect square (sum)} \\
&= (a+b)^2 \quad \text{perfect square (sum) formula} \\
&= \left(\frac{1}{16} + \frac{11}{5}x\right)^2 \quad \text{sub back in } a = \frac{1}{16} \text{ and } b = \frac{11}{5}x.
\end{aligned}$$

Example Factor $16x^4 - 1$.

$$\begin{aligned}
16x^4 - 1 &= \left(4x^2\right)^2 - (1)^2 \ \text{ rewrite to see if it is a difference of squares} \\
&= (a)^2 - (b)^2 \ \text{ Identify as difference of squares, } a = 4x^2,\ b = 1 \\
&= (a+b)(a-b) \ \text{ write down memorized formula} \\
&= (4x^2+1)(4x^2-1) \ \text{ substitute back values for } a = 4x^2 \text{ and } b = 1 \\
&= (4x^2+1)((2x)^2-(1)^2) \ \text{ first polynomial is prime; second is a difference of squares, } a = 2x,\ b = 1 \\
&= (4x^2+1)(a+b)(a-b) \ \text{ write down memorized formula} \\
&= (4x^2+1)(2x+1)(2x-1) \ \text{ substitute back values for } a = 2x \text{ and } b = 1
\end{aligned}$$

Example Factor $\dfrac{1}{16}r^2 - \dfrac{13}{2}rt + 169t^2$.

Notice that the first term and last term look like squares:
$\dfrac{1}{16}r^2 = \left(\dfrac{1}{4}r\right)^2 = a^2$ (so $a = \frac{1}{4}r$) and $169t^2 = (13t)^2 = b^2$ (so $b = 13t$).
Check if the middle term is $2ab = 2\left(\dfrac{1}{4}r\right)(13t) = \dfrac{13}{2}rt$. Yes! So this is a perfect square (difference).

$$\begin{aligned}
\frac{1}{16}r^2 - \frac{13}{2}rt + 169t^2 &= a^2 - 2ab + b^2 \quad \text{identify as a perfect square (difference)} \\
&= (a-b)^2 \quad \text{perfect square (difference) formula} \\
&= \left(\frac{1}{4}r + 13t\right)^2 \quad \text{sub back in } a = \frac{1}{4}r \text{ and } b = 13t.
\end{aligned}$$

A Prime Polynomial is a polynomial that cannot be factored using the techniques we developed here. For now, we are concerned with integers, so something like $x^2 - 3$ would be considered a prime polynomial even though we can write it as $x^2 - 3 = (x - \sqrt{3})(x + \sqrt{3})$. We will certainly want to be factoring even prime polynomials whenever we can we get to the section on polynomial and rational functions.

In practice you aren't told which factoring technique to use, so being able to recognize the different cases is important.