Rational Expressions

A rational expression is a polynomial divided by another polynomial, where the denominator is not the zero polynomial.

Factoring continues to be an important technique when dealing with rational expressions.

The denominator in a rational expression cannot equal zero. We exclude any values that make a denominator zero.

To simplify rational expressions you must separately factor the numerator and denominator, and then cancel using the Basic Rule of Fractions:

$$\frac{a \cdot c}{b \cdot c} = \frac{a}{b} \text{ if } b \neq 0 \text{ and } c \neq 0, \text{ where } a, b \text{ and } c \text{ are polynomials.}$$

$$\frac{x^2 - 1}{x^2 - 3x - 4} = \frac{\cancel{(x+1)}(x-1)}{(x-4)\cancel{(x+1)}} = \frac{x-1}{x-4} \text{ as long as } x + 1 \neq 0.$$

The basic idea of factoring and canceling terms from the numerator and denominator is the same as used for integer fractions. For example, compare the following:

$$\begin{aligned}
\frac{x^2 - 5x + 6}{2x^9 - 18} &= \frac{(x-2)(x-3)}{2(x-3)(x+3)} \text{ factor the polynomials} \\
&= \frac{(x-2)\cancel{(x-3)}}{2\cancel{(x-3)}(x+3)} \text{ cancel between the numerator and denominator} \\
&= \frac{(x-2)}{2(x+3)}, \ \ x - 3 \neq 0
\end{aligned}$$

$$\begin{aligned}
\frac{2310}{440} &= \frac{(2)(3)(5)(7)(11)}{(2)(2)(2)(5)(11)} \text{ factor the integers} \\
&= \frac{\cancel{(2)}(3)\cancel{(5)}(7)\cancel{(11)}}{\cancel{(2)}(2)(2)\cancel{(5)}\cancel{(11)}} \text{ cancel between the numerator and denominator} \\
&= \frac{(3)(7)}{(2)(2)} \\
&= \frac{21}{4}
\end{aligned}$$

To multiply rational expressions, multiply the numerators and denominators, being careful to use parentheses where needed:

$$\begin{aligned}
\frac{x^2 - 3x + 2}{x^2 - 81} \cdot \frac{x - 9}{x^2 + 5x + 6} &= \frac{(x^2 - 3x + 2)(x - 9)}{(x^2 - 81)(x^2 + 5x + 6)} \\
&= \frac{(x-2)(x-1)\cancel{(x-9)}}{\cancel{(x-9)}(x+9)(x+2)(x+3)} \text{ (factor everything you can, then cancel common factors)} \\
&= \frac{(x-2)(x-1)}{(x+9)(x+2)(x+3)}, \quad x - 9 \neq 0
\end{aligned}$$

Compare to the similar technique for integers: $\frac{5}{46} \times \frac{7}{12} = \frac{5 \times 7}{46 \times 12}$.

It is usually best to leave the final expression in its most factored form, but there will be times when you are solving problems when you will then need to multiply everything out. For now, factored is better!

To divide rational expressions, multiply by the reciprocal of the quantity you are dividing by:

$$\begin{aligned}
\frac{x^2+3x+2}{x^2-9} \div \frac{x-3}{x^2+5x+6} &= \frac{x^2+3x+2}{x^2-9} \cdot \frac{x^2+5x+6}{x-3} \\
&= \frac{(x^2+3x+2)(x^2+5x+6)}{(x^2-9)(x-3)} \\
&= \frac{(x+2)(x+1)(x+2)\cancel{(x+3)}}{\cancel{(x+3)}(x-3)(x-3)} \\
&= \frac{(x+2)(x+1)(x+2)}{(x-3)(x-3)}, \quad x+3 \neq 0 \\
&= \frac{(x+2)^2(x+1)}{(x-3)^2}, \quad x+3 \neq 0
\end{aligned}$$

Compare to the similar technique for integers: $\frac{1}{4} \div \frac{7}{12} = \frac{1}{4} \times \frac{12}{7} = \frac{1 \times 12}{4 \times 7}$.

Example Simplify $\frac{x^2+3x-28}{x^2+14x+49} \div \frac{12-3x^2}{x^2+5x-14}$.

We must factor all the polynomials in the expression:

$$\begin{aligned}
x^2+3x-28 &= (x+7)(x-4) \\
x^2+14x+49 &= (x+7)(x+7) \\
12-3x^2 &= 3(4-x^2) = 3(2-x)(2+x) \\
x^2+5x-14 &= (x-2)(x+7)
\end{aligned}$$

$$\begin{aligned}
\frac{x^2+3x-28}{x^2+14x+49} \div \frac{12-3x^2}{x^2+5x-14} &= \frac{x^2+3x-28}{x^2+14x+49} \times \frac{x^2+5x-14}{12-3x^2} \\
&= \frac{(x^2+3x-28)(x^2+5x-14)}{(x^2+14x+49)(12-3x^2)} \\
&= \frac{\cancel{(x+7)}(x-4)(x-2)\cancel{(x+7)}}{\cancel{(x+7)}\cancel{(x+7)}(3)(2-x)(2+x)} \\
&= \frac{(x-4)(x-2)}{3(2-x)(2+x)}, \quad x+7 \neq 0 \\
&= \frac{(x-4)(x-2)}{-3(x-2)(2+x)}, \quad x+7 \neq 0 \\
&= \frac{(x-4)\cancel{(x-2)}}{-3\cancel{(x-2)}(2+x)} \\
&= \frac{(x-4)}{-3(2+x)}, \quad x+7 \neq 0, x-2 \neq 0 \\
&= -\frac{x-4}{3(2+x)}, \quad x \neq -7, 2
\end{aligned}$$

To add or subtract rational expressions, you must find a common denominator for the expressions. This again requires factoring.

1. Factor each denominator completely.
2. The LCD (Lowest Common Denominator) is a product containing each different factor.
3. If a factor occurs more than once, the LCD will contain that factor repeated the greatest number of times it occurs in any one denominator.

Example Simplify $\dfrac{x-8}{x^2-4x+3}+\dfrac{x+2}{x^2-1}$.

Factor everything:

$$\begin{aligned} x^2-4x+3 &= (x-3)(x-1) \\ x^2-1 &= (x-1)(x+1) \end{aligned}$$

Start simplifying:

$$\begin{aligned} \frac{x-8}{x^2-4x+3}+\frac{x+2}{x^2-1} &= \frac{x-8}{(x-3)(x-1)}+\frac{x+2}{(x-1)(x+1)} \\ &= \frac{(x-8)(x+1)}{(x-3)(x-1)(x+1)}+\frac{(x+2)(x-3)}{(x-1)(x+1)(x-3)} \quad \text{(get common denominator)} \\ &= \frac{(x-8)(x+1)+(x+2)(x-3)}{(x-3)(x-1)(x+1)} \quad \text{(simplify)} \\ &= \frac{x^2-7x-8+x^2-x-6}{(x-3)(x-1)(x+1)} \\ &= \frac{2x^2-8x-14}{(x-3)(x-1)(x+1)} \\ &= \frac{2(x^2-4x-7)}{(x-3)(x-1)(x+1)} \quad \text{(numerator is a prime polynomial)} \end{aligned}$$

Example Simplify $\dfrac{x-4}{x^2-4x+3}-\dfrac{x+2}{x^2-1}$.

Factor everything:

$$\begin{aligned} x^2-4x+3 &= (x-3)(x-1) \\ x^2-1 &= (x-1)(x+1) \end{aligned}$$

Start simplifying:

$$\begin{aligned}
\frac{x-4}{x^2-4x+3} - \frac{x+2}{x^2-1} &= \frac{x-4}{(x-3)(x-1)} - \frac{x+2}{(x-1)(x+1)} \\
&= \frac{(x-4)(x+1)}{(x-3)(x-1)(x+1)} - \frac{(x+2)(x-3)}{(x-1)(x+1)(x-3)} \quad \text{(get common denominator)} \\
&= \frac{(x-4)(x+1)-(x+2)(x-3)}{(x-3)(x-1)(x+1)} \quad \text{(simplify)} \\
&= \frac{x^2-3x-4-(x^2-x-6)}{(x-3)(x-1)(x+1)} \quad \text{(be careful to distribute minus sign!)} \\
&= \frac{x^2-3x-4-x^2+x+6}{(x-3)(x-1)(x+1)} \\
&= \frac{-2x+2}{(x-3)(x-1)(x+1)} \\
&= \frac{-2\cancel{(x-1)}}{(x-3)\cancel{(x-1)}(x+1)} \\
&= \frac{-2}{(x-3)(x+1)}, \quad x \neq 1
\end{aligned}$$