Note that on the final exam the focus will be on sketching, not remembering the directrix, focus, etc. formulas. You want to be able to complete the square, find vertex, and know if the parabola opens up, down, left, or right.

Questions

1. Sketch $6x + 3y - x^2 = 9$ by hand. Include all steps in your solution. Identify the focus and directrix of the parabola.

2. Sketch $3y^2 - 4y + 3x - 7 = 0$ by hand. Include all steps in your solution. Identify the focus and directrix of the parabola.

3. Sketch $y^2 - 3y - 3x + 7 = 0$ and $y - x^2 + x = 0$ by hand on the same set of axis. Do the curves intersect? If so, can you determine the points of intersection by hand?

4. Analyze the quadratic $y = ax^2 + bx + c$, $a > 0$, and show that it is a parabola. Determine the vertex, focus, and directrix.

Solutions

1. Sketch $6x + 3y - x^2 = 9$ by hand. Include all steps in your solution. Identify the focus and directrix of the parabola.

$6x+3y-x^2=9$

Complete the square in x.

$-1(x^2-6x) = -3y+9$

$-1(\underbrace{x^2-6x+9}-9) = -3y+9$

$-1([x-3]^2-9) = -3y+9$

$-[x-3]^2 \cancel{+9} = -3y\cancel{+9}$

$(x-3)^2 = 3y$

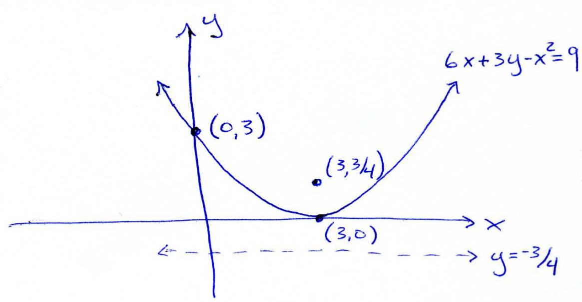
This is standard form of parabola opening up, with ~~center~~ vertex $(3,0)$.

$(x-h)^2 = 4p(y-k)$

Also, $4p=3 \Rightarrow p=3/4$.

Focus is $(3,3/4) = (h,k+p)$

Directrix is $y=-3/4$.



2. Sketch $3y^2 - 4y + 3x - 7 = 0$ by hand. Include all steps in your solution. Identify the focus and directrix of the parabola.

$3y^2 - 4y + 3x - 7 = 0$

complete square in y

$3\left[\underbrace{y^2 - \frac{4}{3}y + \left(\frac{2}{3}\right)^2} - \left(\frac{2}{3}\right)^2\right] + 3x - 7 = 0$

$3\left[\left(y - \frac{2}{3}\right)^2 - \left(\frac{2}{3}\right)^2\right] + 3x - 7 = 0$

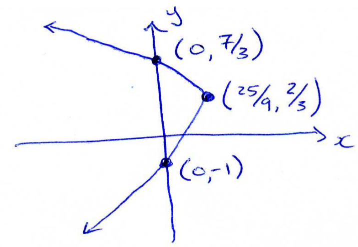
$3\left(y - \frac{2}{3}\right)^2 - \frac{4}{3} + 3x - 7 = 0$

$3\left(y - \frac{2}{3}\right)^2 = -3x + \frac{25}{3}$

$\left(y - \frac{2}{3}\right)^2 = -x + \frac{25}{9}$

$\left(y - \frac{2}{3}\right)^2 = -1\left(x - \frac{25}{9}\right)$

This is the standard form of a parabola opening to the ~~left~~ left with vertex $\left(\frac{25}{9}, \frac{2}{3}\right)$.



If $x = 0$, $\left(y - 2/3\right)^2 = 25/9$

$y = \frac{2}{3} \pm \frac{5}{3} = \frac{7}{3}, -1.$

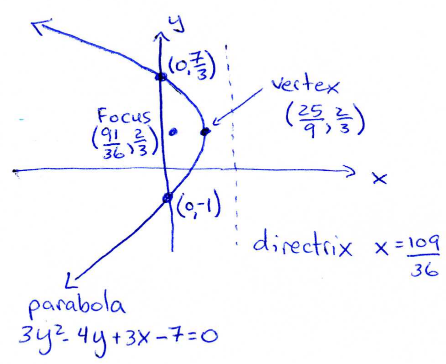
I wanted to get these points $(0, 7/3)$ and $(0,-1)$, to make my sketch more accurate.

The standard form is
$(y-k)^2 = -4p(x-h)$

so $-4p = -1 \Rightarrow p = 1/4$.

The focus is $\left(\frac{25}{9} - \frac{1}{4}, 2/3\right) = \left(\frac{91}{36}, 2/3\right)$

The directrix is $x = \frac{25}{9} + \frac{1}{4} = \frac{109}{36}$



3. Sketch $y^2 - 3y - 3x + 7 = 0$ and $y - x^2 + x = 0$ by hand on the same set of axis. Do the curves intersect? If so, can you determine the points of intersection by hand?

Sketch $y^2 - 3y - 3x + 7 = 0$

$$\underbrace{y^2 - 3y + \left(\tfrac{3}{2}\right)^2} - \left(\tfrac{3}{2}\right)^2 = 3x - 7$$

$$\left(y - \tfrac{3}{2}\right)^2 = 3x - 7 + \frac{9}{4}$$

$$\left(y - \tfrac{3}{2}\right)^2 = 3x - \frac{19}{4}$$

$$\left(y - \tfrac{3}{2}\right)^2 = 3\left(x - \frac{19}{12}\right)$$

parabola, opens right, vertex $\left(\frac{19}{12}, \frac{3}{2}\right)$.

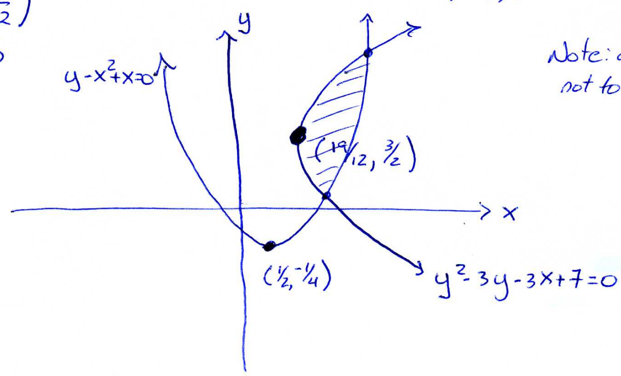
Sketch $y - x^2 + x = 0$

$$x^2 - x = y$$

$$\underbrace{x^2 - x + \tfrac{1}{4}} - \frac{1}{4} = y$$

$$\left(x - \tfrac{1}{2}\right)^2 = y + \frac{1}{4}$$

parabola, opens up, vertex $\left(\frac{1}{2}, -\frac{1}{4}\right)$.



Note: diagram is not to scale.

Points of intersection:

$y - x^2 + x = 0 \to y = x^2 - x$, sub into other equation

$(x^2 - x)^2 - 3(x^2 - x) - 3x + 7 = 0$ which will be difficult to solve.

We would need to use a computer to proceed.

4. Analyze the quadratic $y = ax^2 + bx + c$, $a > 0$, and show that it is a parabola. Determine the vertex, focus, and directrix.

$$y = ax^2 + bx + c$$

$$y - c = a\left(x^2 + \frac{b}{a}x\right)$$

$$= a\left[\underbrace{x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2} - \left(\frac{b}{2a}\right)^2\right]$$

$$= a\left[\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2}\right]$$

$$= a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a}$$

$$y - c + \frac{b^2}{4a} = a\left(x + \frac{b}{2a}\right)^2$$

$$y + \frac{b^2 - 4ac}{4a} = a\left(x + \frac{b}{2a}\right)^2$$

$$\Rightarrow \left(x + \frac{b}{2a}\right)^2 = \frac{1}{a}\left(y + \frac{b^2 - 4ac}{4a}\right)$$

Compare to $(x-h)^2 = 4p(y-k)$

parabola opens up, vertex (h,k).

So if $a>0$, then this is a parabola that opens up with vertex $\left(-\frac{b}{2a}, -\frac{(b^2-4ac)}{4a}\right)$.

Since $4p = \frac{1}{a} \Rightarrow p = \frac{1}{4a}$.

Focus $(h, k+p) = \left(-\frac{b}{2a}, -\frac{(b^2-4ac)}{4a} + \frac{1}{4a}\right)$

~~Directrix~~ $\cancel{(h, k-p) = \left(-\frac{b}{2a}, -\frac{(b^2-4ac)}{4a} - \frac{1}{4a}\right)}$

Aside, earlier in the course, we saw the vertex of $f(x) = ax^2 + bx + c$ was $\left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right)$.

Since $f\left(-\frac{b}{2a}\right) = a\left(-\frac{b}{2a}\right)^2 + b\left(-\frac{b}{2a}\right) + c$

$$= \frac{ab^2}{4a^2} - \frac{b^2}{2a} + c$$

$$= \frac{b^2}{4a} - \frac{2b^2}{4a} + \frac{4ac}{4a}$$

$$= \frac{-b^2 + 4ac}{4a} = -\frac{(b^2 - 4ac)}{4a}$$

this agrees with what we found here.

Whoops! I wasn't thinking.

The directrix is $y = k - p$

$$y = -\frac{(b^2 - 4ac)}{4a} - \frac{1}{4a}$$

$$y = \frac{-1 - b^2 + 4ac}{4a}$$