Note that on the final exam the focus will be on sketching, not remembering the foci, vertices, etc. formulas. You want to be able to complete the square, determine the box the ellipse is inside, and sketch the ellipse.

Questions

1. Sketch $9x^2 + 4y^2 - 18x + 8y - 23 = 0$ by hand. Include all steps in your solution. Identify the vertices and foci of the ellipse.

2. Sketch $x^2/4 + y^2/9 = 1$ and $x^2 + y^2 = 4$ by hand on the same set of axis. Do the curves intersect? If so, can you determine the points of intersection by hand?

3. The graph of the equation $\dfrac{(x-h)^2}{a^2} + \dfrac{(y-k)^2}{b^2} = 0$ is considered to be a degenerate ellipse. Describe the graph.

4. Prove that the nondegenerate graph of the equation $Ax^2 + Cy^2 + Dx + Ey + F = 0$ is an ellipse if $AC > 0$.

5. Determine the perimeter of a triangle with one vertex on the ellipse $x^2/a^2 + y^2/b^2 = 1$ and the other two vertices at the foci of the ellipse.

Solutions

1. Sketch $9x^2 + 4y^2 - 18x + 8y - 23 = 0$ by hand. Include all steps in your solution. Identify the vertices and foci of the ellipse.

$9x^2 + 4y^2 - 18x + 8y - 23 = 0$

complete the square in both x and y.

$9x^2 - 18x + 4y^2 + 8y = 23$

$9[\underbrace{x^2 - 2x + 1} - 1] + 4[\underbrace{y^2 + 2y + 1} - 1] = 23$

$9[(x-1)^2 - 1] + 4[(y+1)^2 - 1] = 23$

$9(x-1)^2 - 9 + 4(y+1)^2 - 4 = 23$

$9(x-1)^2 + 4(y+1)^2 = 36$

$\dfrac{(x-1)^2}{2^2} + \dfrac{(y+1)^2}{3^2} = 1$

ellipse! center: $(1,-1)$.

If $x = 1$, $\dfrac{(y+1)^2}{3^2} = 1$

$y + 1 = \pm 3$

$y = 2, -4$.

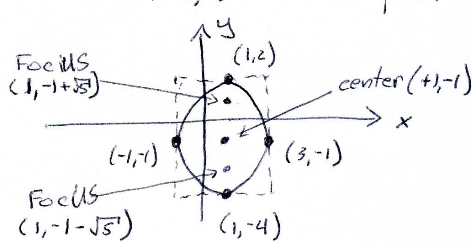
so $(1,2)$, $(1,-4)$ are on ellipse.

If $y = -1$, $\dfrac{(x-1)^2}{2^2} = 1$

$x - 1 = \pm 2$

$x = 3, -1$

so $(-1,-1)$, $(3,-1)$ are on ellipse.



Since $c^2 = 3^2 - 2^2 = 5$, $c = \sqrt{5}$.

Vertices are ends of ellipse along stretched direction (major axis), which are $(1,2)$, $(1,-4)$.

2. Sketch $x^2/4 + y^2/9 = 1$ and $x^2 + y^2 = 4$ by hand on the same set of axis. Do the curves intersect? If so, can you determine the points of intersection by hand?

Sketch $\frac{x^2}{4} + \frac{y^2}{9} = 1$

ellipse → get the box that bounds it!

If $x=0$, $\frac{y^2}{9} = 1$

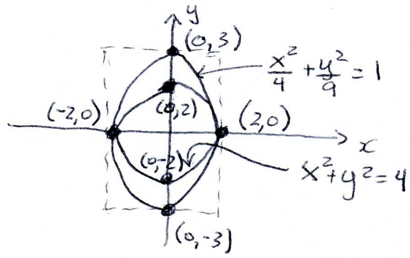
$y = \pm 3$

$(0,3)$, $(0,-3)$ on box

If $y=0$, $\frac{x^2}{4} = 1$

$x = \pm 2$

$(2,0)$, $(-2,0)$ on box



Sketch $x^2 + y^2 = 4$

$x^2 + y^2 = 2^2$

circle, center $(0,0)$ radius 2.

Verify intersection is $(\pm 2, 0)$:

Solve $\left.\begin{array}{l} \frac{x^2}{4} + \frac{y^2}{9} = 1 \\ x^2 + y^2 = 4 \end{array}\right\}$ system of 2 equations in 2 unknowns.

From second equation, $x^2 = 4 - y^2$, sub into first equation:

$\frac{4-y^2}{4} + \frac{y^2}{9} = 1$

$1 - \frac{y^2}{4} + \frac{y^2}{9} = 1$

$\rightarrow y = 0$.

→ If $y=0$, then $x^2 + 0^2 = 4$

$x = \pm 2$.

Only two points of intersection, $(2,0)$, and $(-2,0)$.

3. The graph of the equation $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 0$ is considered to be a degenerate ellipse. Describe the graph.

If $x = h$, then $\frac{(y-k)^2}{b^2} = 0$

$y = k$.

point (h,k).

If $y = k$, then $\frac{(x-h)^2}{a^2} = 0$

$x = h$.

point (h,k).

So it looks like $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 0$ consists of a single point, (h,k), the center of the ellipse $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$.

→ You can also see this from trying to solve directly for y:

$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 0$

$(y-k)^2 = -\frac{b^2}{a^2}(x-h)^2$

$y - k = \pm\sqrt{-\frac{b^2}{a^2}(x-h)^2}$

$= \frac{b}{a}(x-h)i$

$y = k + \frac{b}{a}(x-h)i$.

So the only real valued solution is when $x = h$, where $y = k$. The point (h,k)

4. Prove that the nondegenerate graph of the equation $Ax^2 + Cy^2 + Dx + Ey + F = 0$ is an ellipse if $AC > 0$.

$Ax^2 + Cy^2 + Dx + Ey + F = 0$

complete the square in x and y.

$$A\Big[\underbrace{x^2 + \tfrac{D}{A}x + \left(\tfrac{D}{2A}\right)^2}\, - \left(\tfrac{D}{2A}\right)^2\Big] + C\Big[\underbrace{y^2 + \tfrac{E}{C}y + \left(\tfrac{E}{2C}\right)^2}\, - \left(\tfrac{E}{2C}\right)^2\Big] = -F$$

$$A\left[\left(x + \tfrac{D}{2A}\right)^2 - \left(\tfrac{D}{2A}\right)^2\right] + C\left[\left(y + \tfrac{E}{2C}\right)^2 - \left(\tfrac{E}{2C}\right)^2\right] = -F$$

$$A\left(x + \tfrac{D}{2A}\right)^2 - \frac{D^2}{4A} + C\left(y + \tfrac{E}{2C}\right)^2 - \frac{E^2}{4C} = -F$$

$$A\left(x + \tfrac{D}{2A}\right)^2 + C\left(y + \tfrac{E}{2C}\right)^2 = \frac{D^2}{4A} + \frac{E^2}{4C} - F$$

$$\frac{\left(x + \frac{D}{2A}\right)^2}{C} + \frac{\left(y + \frac{E}{2C}\right)^2}{A} = \frac{D^2}{4A^2C} + \frac{E^2}{4AC^2} - \frac{F}{AC}$$

$$\frac{\left(x + \frac{D}{2A}\right)^2}{C} + \frac{\left(y + \frac{E}{2C}\right)^2}{A} = \frac{D^2C + E^2A - 4ACF}{4A^2C^2}$$

⇑ we need this to be 1 for an ellipse.

→ what we are concerned about are the signs of the factors. Let $W = \dfrac{D^2C + E^2A - 4ACF}{4A^2C^2}$,

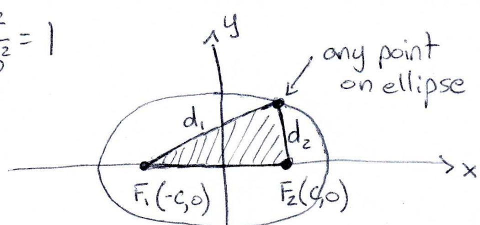
then $$\frac{\left(x + \frac{D}{2A}\right)^2}{C \cdot W} + \frac{\left(y + \frac{E}{2C}\right)^2}{A \cdot W} = 1.$$

If $C > 0, A > 0, W > 0$ this is an ellipse.

Also an ellipse if $C < 0, A < 0, W < 0$.

5. Determine the perimeter of a triangle with one vertex on the ellipse $x^2/a^2 + y^2/b^2 = 1$ and the other two vertices at the foci of the ellipse.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$



By definition, $d_1 + d_2 = 2a$.

(see our derivation, we included the 2 to simplify things later)

$F_1F_2 = 2c$.

So Perimeter of triangle is $2a + 2c$.