Note that on the final exam the focus will be on sketching, not remembering the foci, vertices, etc. formulas. You want to be able to complete the square, determine the box the hyperbola is outside, and if it opens left/right or up/down.

Questions

1. Sketch $9x^2 - 4y^2 - 36x + 8y - 4 = 0$ by hand. Include all steps in your solution. Identify the center, vertices and foci of the hyperbola.

2. Prove that for the hyperbola $x^2/a^2 - y^2/b^2 = 1$ if $x = c$, then $y = \pm b^2/a$. Why is it reasonable to define the *focal width* of such hyperbolas to be $2b^2/a$?

3. The graph of the equation $\dfrac{(x-h)^2}{a^2} - \dfrac{(y-k)^2}{b^2} = 0$ is considered to be a degenerate hyperbola. Describe the graph.

4. A comet following a hyperbolic path about the Sun has a perihelion of 120 Gm. When the line from the comet to the Sun is perpendicular to the focal axis of the orbit, the comet is 250 Gm from the Sun. Sketch the trajectory by hand, and determine the values of a, b, and c. What are the coordinates of the center of the Sun if the center of the hyperbolic orbit is $(0, 0)$ and the Sun lies on the positive x-axis?

Solutions

1. Sketch $9x^2 - 4y^2 - 36x + 8y - 4 = 0$ by hand. Include all steps in your solution. Identify the center, vertices and foci of the hyperbola.

$9x^2 - 4y^2 - 36x + 8y - 4 = 0$

complete square in x and y.

$9[x^2 - 4x + 4 - 4] - 4[y^2 + 2y + 1 - 1] = 4$

$9[(x-2)^2 - 4] - 4[(y+1)^2 - 1] = 4$

$9(x-2)^2 - 36 - 4(y+1)^2 + 4 = 4$

$9(x-2)^2 - 4(y+1)^2 = 36$

$\dfrac{(x-2)^2}{2^2} - \dfrac{(y+1)^2}{3^2} = 1.$

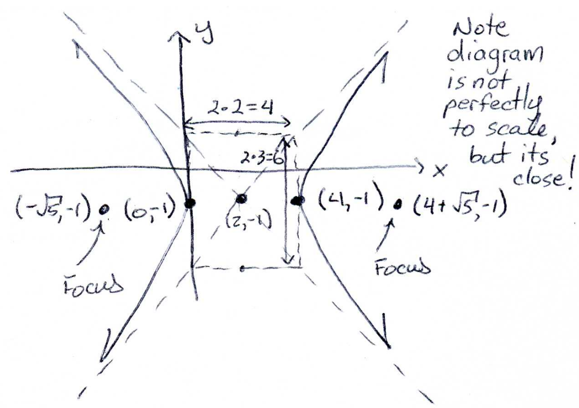
hyperbola! Center $(2,-1)$.

Get box: If $y = -1$, then $\dfrac{(x-2)^2}{4} = 1$

$x - 2 = \pm 2$

$x = 4, 0.$

So $(0,-1)$ and $(4,-1)$ are on the ellipse. This helps us see the hyperbola opens left/right, since it touches box on left/right sides.



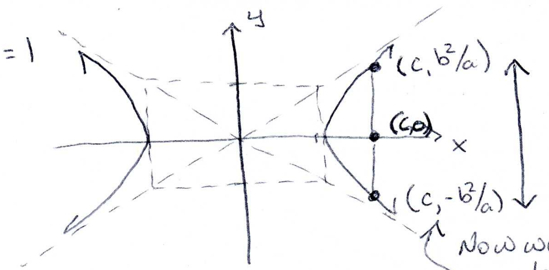
Now $c^2 = 3^2 - 2^2 = 5 \Rightarrow c = \sqrt{5}$!

Focus: $(-\sqrt{5}, -1)$

$(4 + \sqrt{5}, -1)$.

2. Prove that for the hyperbola $x^2/a^2 - y^2/b^2 = 1$ if $x = c$, then $y = \pm b^2/a$. Why is it reasonable to define the *focal width* of such hyperbolas to be $2b^2/a$?

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$



this distance is $2b^2/a$, so we call that the focal width.

Now we can label these points

If $x = c$, then $\frac{c^2}{a^2} - \frac{y^2}{b^2} = 1$

$$y^2 = b^2\left(\frac{c^2}{a^2} - 1\right)$$

use the fact that $c^2 = a^2 + b^2$:

$$y^2 = b^2\left(\frac{a^2+b^2}{a^2} - 1\right)$$

$$y^2 = \frac{b^4}{a^2}$$

$$\rightarrow \quad y = \pm\frac{b^2}{a}$$

3. The graph of the equation $\dfrac{(x-h)^2}{a^2} - \dfrac{(y-k)^2}{b^2} = 0$ is considered to be a degenerate hyperbola. Describe the graph.

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 0$$

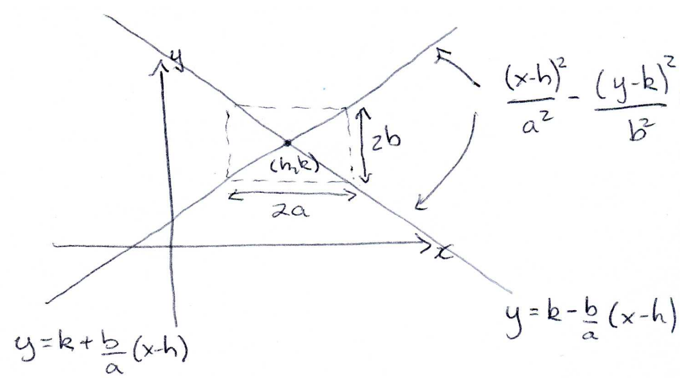
$$\frac{(x-h)^2}{a^2} = \frac{(y-k)^2}{b^2}$$

$$\pm\frac{x-h}{a} = \frac{y-k}{b}$$

$$y = k \pm \frac{b}{a}(x-h)$$

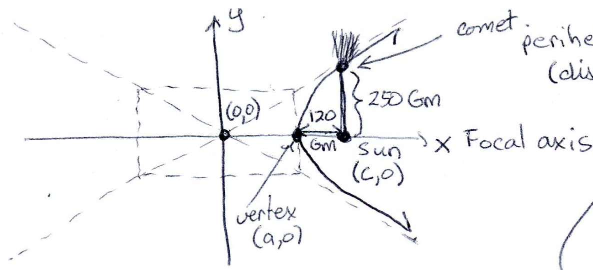
These are just the asymptote lines for the hyperbola

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1.$$



$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 0$$

4. A comet following a hyperbolic path about the Sun has a perihelion of 120 Gm. When the line from the comet to the Sun is perpendicular to the focal axis of the orbit, the comet is 250 Gm from the Sun. Sketch the trajectory by hand, and determine the values of a, b, and c. What are the coordinates of the center of the Sun if the center of the hyperbolic orbit is $(0, 0)$ and the Sun lies on the positive x-axis?



perihelion is 120 Gm
(distance of closest approach)

Comet path is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, $x > 0$.

From diagram, $c - a = 120$

when $x = c$, $\frac{c^2}{a^2} - \frac{y^2}{b^2} = 1 \rightarrow y^2 = b^2\left(\frac{c^2}{a^2} - 1\right)$

$$250^2 = b^2\left(\frac{c^2}{a^2} - 1\right)$$

We have two equations, but three unknowns: a, b, c.
Another equation is $c^2 = a^2 + b^2$.

$c - a = 120$ (1)

$b^2\left(\frac{c^2}{a^2} - 1\right) = 250^2$ (2)

$c^2 = a^2 + b^2$. (3)

Put (3) into (2):

$$b^2\left(\frac{a^2+b^2}{a^2} - 1\right) = 250^2$$

$$\frac{b^4}{a^2} = 250^2$$

$$b^2 = 250a$$

So (3) becomes

$$c^2 = a^2 + 250a.$$

We now have 2 equations in 2 unknowns:

$$\left.\begin{array}{l} c - a = 120 \\ c^2 = a^2 + 250a \end{array}\right\}$$

Put first equation, $c = 120 + a$, into second

$$(120 + a)^2 = a^2 + 250a$$

$$120^2 + 240a + \cancel{a^2} = \cancel{a^2} + 250a$$

$$120^2 = 10a$$

$$\rightarrow a = 1440 \text{ Gm}$$

Almost there!

$$b = \sqrt{250a} = \sqrt{250 \cdot 1440} = 600 \text{ Gm}$$

$$c = 120 + a = 1560 \text{ Gm}$$

The sun is located at $(c,0) = (1560,0)$.