Questions

1. Given the parametric equations

$$x = t + 1, \quad y = t^2 - 2t.$$

Find the points determined by $t = -3, -2, -1, 0, 1, 2, 3$.
Find a direct relationship between x and y. Is this an explicit function relationship?
Graph the relationship in the xy-plane.

2. Find a formula for $f^{-1}(x)$ given $f(x) = \sqrt{x-3}$. Give the domain of $f^{-1}(x)$.

3. Find a formula for $f^{-1}(x)$ given $f(x) = (x-2)^{1/3}$. Give the domain of $f^{-1}(x)$.

4. Find a formula $f^{-1}(x)$ for the inverse of the function (you do not have to discuss domain and range):

$$f(x) = \frac{1+5x}{3-2x}$$

5. Find a formula $f^{-1}(x)$ for the inverse of the function $f(x) = -\sqrt{2x-1} + 1$. What are the domain and range for f? The domain and range for f^{-1}?

Solutions

1. Given the parametric equations

$$x = t + 1, \quad y = t^2 - 2t.$$

Find the points determined by $t = -3, -2, -1, 0, 1, 2, 3$.
Find a direct relationship between x and y. Is this an explicit function relationship?
Graph the relationship in the xy-plane.

t	$x = t + 1$	$y = t^2 - 2t$
-3	-2	15
-2	-1	8
-1	0	3
0	1	0
1	2	-1
2	3	0
3	4	3

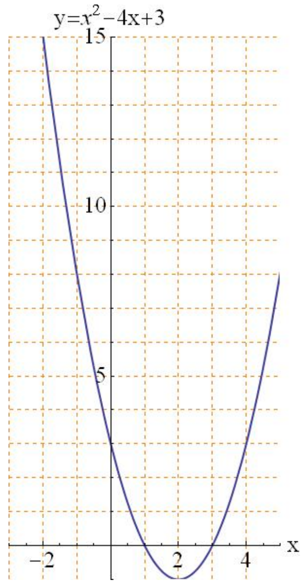
To find a direct relationship, we need to eliminate the parameter t from the two equations.

Solve $x = t + 1$ for t: $t = x - 1$.
Substitute $t = x - 1$ into the equation $y = t^2 - 2t$:

$$\begin{aligned} y &= t^2 - 2t \\ &= (x-1)^2 - 2(x-1) \\ &= x^2 - 2x + 1 - 2x + 2 \\ &= x^2 - 4x + 3 \end{aligned}$$

The function is a quadratic. We could sketch this by hand, but here I want to check that the sketch agrees with the table of values we found above, so I will use a computer to sketch it for me.

Here is a sketch I created using *Mathematica*:



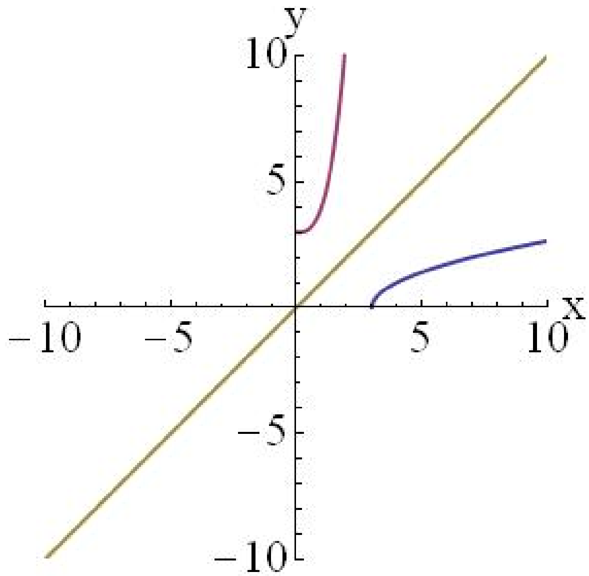
Notice the points on the sketch agree with the table we created above.

2. Find a formula for $f^{-1}(x)$ given $f(x) = \sqrt{x-3}$. Give the domain of $f^{-1}(x)$.

$$\begin{aligned} y &= \sqrt{x-3} && \text{Step 1: let } y = f(x) \\ x &= \sqrt{y-3} && \text{Step 2: Flip } x \text{ and } y \\ x^2 &= (y-3) && \text{Step 3: Solve for } y \\ x^2+3 &= y \\ y &= x^2+3 \\ f^{-1}(x) &= x^2+3 && \text{Finally, } f^{-1}(x) = y \end{aligned}$$

The function f has a range of $y \in [0, \infty)$. This imposes a restriction on the domain of $f^{-1}(x)$. Since x^2+3 has domain $x \in (-\infty, \infty)$, the domain of $f^{-1}(x)$ is $x \in [0, \infty)$.

It wasn't asked for in the question, but here is a sketch I created using *Mathematica*, that shows the graphical reflection of the inverse function about the lines $y = x$.



3. Find a formula for $f^{-1}(x)$ given $f(x) = (x-2)^{1/3}$. Give the domain of $f^{-1}(x)$.

$$\begin{aligned}
y &= (x-2)^{1/3} \quad \text{Step 1: let } y = f(x) \\
x &= (y-2)^{1/3} \quad \text{Step 2: Flip } x \text{ and } y \\
x^3 &= (y-2) \quad \text{Step 3: Solve for } y \\
x^3+2 &= y \\
y &= x^3+2 \\
f^{-1}(x) &= x^3+2 \quad \text{Finally, } f^{-1}(x) = y
\end{aligned}$$

Since there is no restriction imposed by f, the domain of $f^{-1}(x)$ is $x \in (-\infty, \infty)$.

4. Find a formula $f^{-1}(x)$ for the inverse of the function (you do not have to discuss domain and range):

$$f(x) = \frac{1+5x}{3-2x}$$

$$\begin{aligned}
y &= \frac{1+5x}{3-2x} \quad \text{Step 1: let } y = f(x) \\
x &= \frac{1+5y}{3-2y} \quad \text{Step 2: Flip } x \text{ and } y \\
(3-2y)x &= 1+5y \\
3x-2xy &= 1+5y \\
-2xy-5y &= 1-3x \\
(-2x-5)y &= 1-3x \\
y &= \frac{1-3x}{-2x-5} \quad = \frac{3x-1}{2x+5} \\
f^{-1}(x) &= \frac{3x-1}{2x+5} \quad \text{Finally, we have } f^{-1}(x) = y = \frac{3x-1}{2x+5}
\end{aligned}$$

5. Find a formula $f^{-1}(x)$ for the inverse of the function $f(x) = -\sqrt{2x-1}+1$. What are the domain and range for f? The domain and range for f^{-1}?

Let's do the domain and ranges of f first. Since we must take the square root of something greater than or equal to zero, we require

$$2x - 1 \geq 0 \Rightarrow x \geq \frac{1}{2}$$

The domain of f is $x \geq 1/2$.

Since the square root is always less than or equal to zero (since we have a minus in front), and then we are adding one, the range of f is $y \in (-\infty, 1]$.

These are simply flipped for the inverse function.

The domain of f^{-1} is $x \in (-\infty, 1]$, and the range of f^{-1} is $y \geq 1/2$.

$$\begin{aligned}
y &= -\sqrt{2x-1}+1 \quad \text{Step 1: let } y = f(x) \\
x &= -\sqrt{2y-1}+1 \quad \text{Step 2: Flip } x \text{ and } y \\
(x-1)^2 &= 2y-1 \\
2y &= (x-1)^2+1 \\
y &= \frac{(x-1)^2+1}{2} \\
f^{-1}(x) &= \frac{(x-1)^2+1}{2} \quad \text{Finally, we have } f^{-1}(x) = y = \frac{(x-1)^2+1}{2}
\end{aligned}$$

Summary:

$$f(x) = -\sqrt{2x-1}+1 \text{ with domain } x \in (1/2, \infty), \text{ range } y \in (-\infty, 1).$$

$$f^{-1}(x) = \frac{(x-1)^2+1}{2} \text{ with domain } x \in (-\infty, 1), \text{ range } y \in (1/2, \infty).$$