Solving Systems of Equations

1. Solve the system:

$$x^2 = y$$
$$y^2 = 8x$$

2. Solve the system:

$$4x + 2y = 4$$
$$3x + y = 4$$

3. Solve the system:

$$9x + 2y = 2$$
$$3x + 5y = 5$$

4. Solve the system of equations algebraically

$$\begin{aligned} y^2 &= 4x + 9 \\ y &= -|x| \end{aligned}$$

Draw a well labeled sketch of the situation (the sketch can be drawn without using a calculator).

5. Solve the system algebraically:

$$0.2x = 0.1y - 1.2$$
$$2x - y = 6$$

6. Is it possible to construct a parallelogram with acute angle $\pi/3$ radians, area of 9 cm^2, and perimeter of 12 cm? Justify your answer using appropriate mathematics, and explain each step of your solution using English as well as math.

7. Solve the system:

$$x^2 - y^2 = 1$$
$$x + ay = 1$$

Note: Your solution with involve the unspecified constant a.

Solving Systems of Inequalities Graphically

8. Graph by hand the solution to the system of inequalities, and determine the points of intersection for the region.

$$x^2 + y^2 \leq 9$$
$$-x + y^2 - 1 \leq 0$$

9. Graph by hand the solution to the system of inequalities:

$$5x - 3y > 1$$
$$3x + 4y \leq 18$$

10. Graph by hand the solution to the system of inequalities, and determine the points of intersection for the region.

$$\begin{aligned} 2x + y &\le 80 \\ x + 2y &\le 80 \\ x &\ge 0 \\ y &\ge 0 \end{aligned}$$

11. Graph by hand the solution to the system of inequalities, and determine the points of intersection for the region. As best you can, describe the region in terms of the values y takes as x varies across the region (this will be an important skill in Calculus II).

$$\begin{aligned} x^2 + y^2 &\le 4 \\ y &\ge |x| \end{aligned}$$

12. Graph by hand the solution to the system of inequalities, and determine the points of intersection for the region. As best you can, describe the region in terms of the values y takes as x varies across the region.

$$\begin{aligned} y &\ge 4x - x^2 \\ y &\le 8x - 2x^2 \end{aligned}$$

Solutions

1. Solve the system algebraically:

$$x^2 = y$$
$$y^2 = 8x$$

$x^2 = y$ (1)
$y^2 = 8x$ (2)

From (2), $y = \pm\sqrt{8x}$. Substitute this into (1):

$x^2 = \pm\sqrt{8x}$ Solve for x.

$x^4 = 8x$ square both sides.

$x^4 - 8x = 0$ factor.

$x(x^3 - 8) = 0$

$x = 0$ or $x^3 - 8 = 0$

$x = (8)^{1/3} = 2$ is only real solution.

If $x = 0$, from (1) $y = x^2 = 0^2 = 0$. So $(0,0)$ is a solution

If $x = 2$, from (1) $y = x^2 = 2^2 = 4$. So $(2,4)$ is a solution.

2. Let's use the substitution method.

From the second equation, we can solve for $y = 4 - 3x$. Substitute this into the first equation:

$$\begin{aligned} 4x + 2y &= 4 \\ 4x + 2(4 - 3x) &= 4 \text{ now, solve for } x \\ 4x + 8 - 6x &= 4 \\ -2x &= 4 - 8 \\ -2x &= -4 \\ x &= 2 \end{aligned}$$

Now, use this value of x in $y = 4 - 3x$ to determine y:

$$\begin{aligned} y &= 4 - 3x \\ y &= 4 - 3(2) \\ y &= -2 \end{aligned}$$

The solution to the system is the ordered pair $(2, -2)$.

3. Let's use the elimination method.

Multiply the second equation by -3 to make the coefficient of x the same in both equations, but with opposite sign.

$$\begin{aligned} 9x + 2y &= 2 \\ -9x - 15y &= -15 \end{aligned}$$

Now add the two equations to eliminate the x (since $9x - 9x = 0$):

$$\begin{aligned} 9x + 2y &= 2 \\ -9x - 15y &= -15 \end{aligned}$$

Adding:

$$\begin{aligned} 2y - 15y &= 2 - 15 \text{ now solve for } y \\ -13y &= -13 \\ y &= 1 \end{aligned}$$

Now, use this value of y in any of the earlier equations to determine x:

$$\begin{aligned} 9x + 2y &= 2 \\ 9x + 2(1) &= 2 \\ 9x + 2 &= 2 \\ 9x &= 0 \\ x &= 0 \end{aligned}$$

The solution to the system is the ordered pair $(0, 1)$.

4. Solve the system of equations

$$y^2 = 4x + 9 \qquad (1)$$
$$y = -|x| \qquad (2)$$

Draw a well labeled sketch of the situation (the sketch can be drawn without using a calculator).

Rewrite Eq. (2) as $y^2 = (-|x|)^2 = x^2$ and substitute into Eq. (1):

$$\begin{aligned} y^2 &= 4x + 9 \\ x^2 &= 4x + 9 \\ x^2 - 4x - 9 &= 0 \\ x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{4 \pm \sqrt{16 + 36}}{2} = 2 \pm \sqrt{13} \end{aligned}$$

For $x = 2 + \sqrt{13}$:

$$y = -|x| = -|2 + \sqrt{13}| = -2 - \sqrt{13}$$

For $x = 2 - \sqrt{13}$:

$$y = -|x| = -|2 - \sqrt{13}| = -(-(2 - \sqrt{13})) = 2 - \sqrt{13}$$

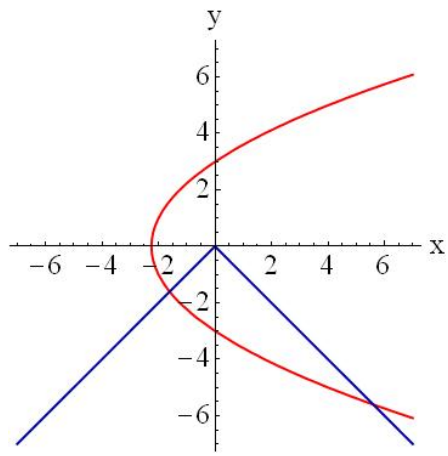
The two solutions are $(x, y) = (2 + \sqrt{13}, -2 - \sqrt{13})$ and $(x, y) = (2 - \sqrt{13}, 2 - \sqrt{13})$

Here is a sketch. This sketch can be drawn by hand, without the aid of a calculator.

$y = -|x|$: This is simply $y = |x|$ reflected about the x axis.

$y^2 = 4x + 9$: This is a parabola. If $x = 0$, $y = \pm 3$. If $y = 0$, $x = -9/4$. Therefore, parabola opens to right.

We see there are two points of intersection of the curves. The red curve is Eq. (1) and the blue curve is Eq. (2).



5. Let's use the elimination method.

Multiply the first equation by -10 to make the coefficient of x the same in both equations, but with opposite sign.

$$\begin{aligned} -2x &= -y + 12 \\ 2x - y &= 6 \end{aligned}$$

Now add the two equations to eliminate the x (since $-2x + 2x = 0$):

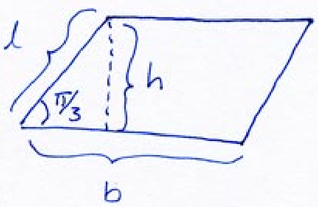
$$\begin{aligned} -y &= -y + 12 + 6 \\ 0 &= 18 \end{aligned}$$

You might think you've made a mistake, but you just need to interpret what you've found.

Since 0 can never equal 18, there is no solution to the system of equations. Graphically, the two equations represent two parallel lines.

6. Is it possible to construct a parallelogram with acute angle $\pi/3$ radians, area of 9 cm^2, and perimeter of 12 cm? Justify your answer using appropriate mathematics, and explain each step of your solution using English as well as math.

Solution: We need to start with a sketch, and bring in the formula for the area of a parallelogram before we can proceed.



Area of parallelogram $= bh$

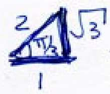
Perimeter of parallelogram $= 2b + 2l$

We will want $\begin{cases} bh = 9 \\ 2b + 2l = 12 \end{cases}$ (1)

The two equations in (1) have three unknowns, b, h, l. We therefore need to eliminate one variable so we will have two equations in two unknowns which we can then solve.

We haven't used the $\pi/3$ yet, so we should be able to bring that in and eliminate a variable.

From the sketch, we see $\sin(\pi/3) = \frac{h}{l}$. We also know $\sin(\pi/3) = \frac{\sqrt{3}}{2}$ from



So $\frac{\sqrt{3}}{2} = \frac{h}{l}$, and we can write $l = \frac{2h}{\sqrt{3}}$. Our Equations (1) now become two equations in two unknowns:

$$\left.\begin{aligned} bh &= 9 \\ 2b + \frac{4h}{\sqrt{3}} &= 12 \end{aligned}\right\} \quad (2)$$

From first equation, $h = \frac{9}{b}$, and subing into the second equation we have $2b + \frac{36}{\sqrt{3}\,b} = 12 \longrightarrow 2b^2 - 12b + \frac{36}{\sqrt{3}} = 0$. This is quadratic in b, so we can use the quadratic formula to solve.

$$b = \frac{-\bar{b} \pm \sqrt{\bar{b}^2 - 4\bar{a}\bar{c}}}{2\bar{a}}$$

(use bars to avoid notational confusion).

Note $\bar{b}^2 - 4\bar{a}\bar{c} = (-12)^2 - 4(2)\left(\frac{36}{\sqrt{3}}\right)$
$= 144 - 96\sqrt{3}$
$\simeq -22.2769$.

Since this is negative, there are no real valued solutions b, and parallelogram with acute angle $\pi/3$, area 9 and perimeter 12 cannot exist!

7. Solve the system:

$$x^2 - y^2 = 1$$
$$x + ay = 1$$

Note: Your solution with involve the unspecified constant a.

$x^2 - y^2 = 1 \quad (1)$

$x + ay = 1 \quad (2)$

Solve (2) for $x = 1 - ay$, and substitute this into (1):

$(1-ay)^2 - y^2 = 1$ solve for y.

$1 - 2ay + a^2y^2 - y^2 = 1$ cancel the 1 (the fact this happened actually makes the problem easier to solve)

$-2ay + a^2y^2 - y^2 = 0$

Factor $y(-2a + a^2y - y) = 0$

$y = 0$ or $-2a + a^2y - y = 0$

$(a^2-1)y = 2a$

$y = \frac{2a}{a^2-1}$.

If $y=0$, then $x = 1 - ay = 1 - a(0) = 1$. So $(1,0)$ is a solution

If $y = \frac{2a}{a^2-1}$, then $x = 1 - ay = 1 - a\left(\frac{2a}{a^2-1}\right)$

$= \frac{a^2-1-2a^2}{a^2-1}$

$= \frac{-1-a^2}{a^2-1}$

$= \frac{1+a^2}{1-a^2}$ So $\left(\frac{1+a^2}{1-a^2}, \frac{2a}{a^2-1}\right)$ is a solution.

8. Graph by hand the solution to the system of inequalities, and determine the points of intersection for the region.

$$x^2 + y^2 < 9$$
$$-x + y^2 - 1 \leq 0$$

Sketch $-x+y^2-1=0$:

$y^2 = x+1$ is a quadratic that opens to the right, shifted left 1 unit.

Test Point: $(0,0)$
$-x+y^2-1 \leq 0$
$-(0)+(0)^2-1 \leq 0$
$-1 \leq 0$ True!
shade side of line with $(0,0)$.

Sketch $x^2+y^2=9$:

$x^2+y^2=3^2$
circle of radius 3 centered at the origin.

Test Point $(0,0)$
$(0)^2+(0)^2 < 9$
$0 < 9$ True!
shade inside of circle

Points of intersection:

Solve $\left.\begin{aligned} y^2 &= x+1 \\ x^2+y^2 &= 9 \end{aligned}\right\}$ system of equations.

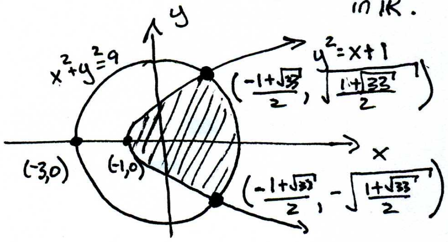
$\Rightarrow x^2+x+1=9$
$x^2+x-8=0$
$x = \dfrac{-1 \pm \sqrt{1-4(1)(-8)}}{2(1)} = \dfrac{-1 \pm \sqrt{33}}{2}$

Use $y^2 = x+1$ to find (x,y) pairs:

If $x = \dfrac{-1+\sqrt{33}}{2}$, $y^2 = \dfrac{-1+\sqrt{33}}{2}+1$
$y^2 = \dfrac{1+\sqrt{33}}{2}$
$y = \pm\sqrt{\dfrac{1+\sqrt{33}}{2}}$

Two points of intersection:
$\left(\dfrac{-1+\sqrt{33}}{2}, +\sqrt{\dfrac{1+\sqrt{33}}{2}}\right)$
$\left(\dfrac{-1+\sqrt{33}}{2}, -\sqrt{\dfrac{1+\sqrt{33}}{2}}\right)$

If $x = \dfrac{-1-\sqrt{33}}{2}$, $y^2 = \dfrac{1-\sqrt{33}}{2} < 0$ no y values in $\mathbb{R}$.



9. Graph by hand the solution to the system of inequalities:

$$5x - 3y > 1$$
$$3x + 4y \leq 18$$

sketch $5x-3y=1$:

If $x=0$, then $-3y=1$
$y = -1/3$
$\Rightarrow (0,-1/3)$.

If $y=0$, then $5x=1$
$x = 1/5$
$\Rightarrow (1/5, 0)$.

Dashed line since we want strictly greater than: $5x-3y>1$.

Test Point is $(0,0)$:
$5(0)-3(0) > 1$
$0 > 1$ False!

shade side of line that does not contain $(0,0)$.

sketch $3x+4y = 18$:

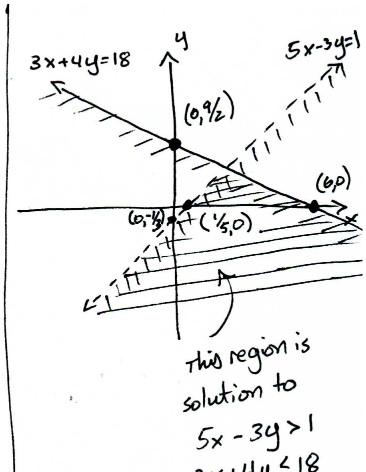
If $x=0$, then $4y=18$
$y = 9/2$
$\Rightarrow (0, 9/2)$

If $y=0$, then $3x=18$
$x=6$
$\Rightarrow (6,0)$.

Solid line since we want less than or equal, $3x+4y \leq 18$.

Test Point is $(0,0)$:
$3(0)+4(0) \leq 18$
$0 \leq 18$ True!

shade side of line that contains $(0,0)$.



10. Graph by hand the solution to the system of inequalities, and determine the points of intersection for the region.

$$\begin{aligned} 2x + y &\le 80 \\ x + 2y &\le 80 \\ x &\ge 0 \\ y &\ge 0 \end{aligned}$$

<u>Sketch $2x+y=80$:</u>

If $x=0$, then $y=80$.

$\Rightarrow (0,80)$.

If $y=0$, then $2x=80$

$x=40$

$\Rightarrow (40,0)$.

Solid line since

$2x+y \le 80$.

Test Point is $(0,0)$:

$2(0)+(0) \le 80$

$0 \le 80$ True!

shade side of line containing $(0,0)$.

<u>sketch $x+2y=80$:</u>

If $x=0$, then $2y=80$

$y=40$

$\Rightarrow (0,40)$.

If $y=0$, then $x=80$

$\Rightarrow (80,0)$.

Solid line since

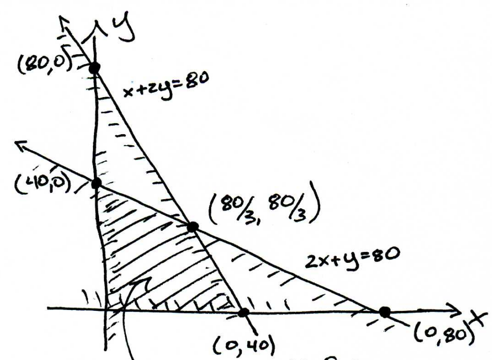
$x+2y \le 80$.

Test Point is $(0,0)$:

$(0)+2(0) \le 80$

$0 \le 80$ True!

shade side of line containing $(0,0)$.



region satisfying

$2x+y \le 80$

$x+2y \le 80$

$x \ge 0$

$y \ge 0$

intersection point: $2x+4y=160$ (multiply equation by 2)

$\underline{2x+y=80}$ (subtract other equation)

$3y=80$

$\Rightarrow y=\frac{80}{3}$. Then $x=80-2y=80-2\left(\frac{80}{3}\right)=\frac{80}{3}$

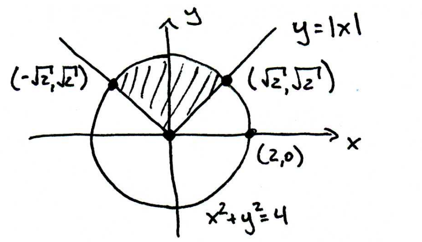
$\Rightarrow$ intersection is at $(80/3, 80/3)$

11. Graph by hand the solution to the system of inequalities, and determine the points of intersection for the region. As best you can, describe the region in terms of the values y takes as x varies across the region (this will be an important skill in Calculus II).

$$x^2 + y^2 \le 4$$
$$y \ge |x|$$

Sketch $x^2+y^2=4$:
Circle radius 2
centered at origin.
Test Point $(0,0)$:
$(0)^2+(0)^2 \le 4$
$0 \le 4$ True!
shade inside of circle

sketch $y=|x|$
(one of basic functions)
Test Point $(0,1)$:
$1 \ge |0|$ True!
shade side of line
containining $(0,1)$



Intersection: $\left.\begin{matrix} x^2+y^2=4 \\ y=|x| \end{matrix}\right\}$ system of equations.

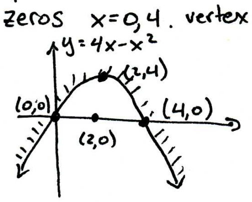
$x^2+(|x|)^2=4$
$2x^2=4$
$x^2=2$
$x=\pm\sqrt{2}$
If $x=+\sqrt{2}$: $y=|\sqrt{2}|=\sqrt{2} \Rightarrow (\sqrt{2},\sqrt{2})$.
$x=-\sqrt{2}$: $y=|-\sqrt{2}|=\sqrt{2} \Rightarrow (-\sqrt{2},\sqrt{2})$.

Description of region:
for $x \in [-\sqrt{2},\sqrt{2}]$,
y varies from
$y=|x|$ to $y=\sqrt{4-x^2}$.

12. Graph by hand the solution to the system of inequalities, and determine the points of intersection for the region. As best you can, describe the region in terms of the values y takes as x varies across the region.

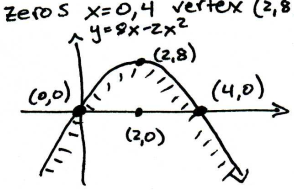
$$y \ge 4x - x^2$$
$$y \le 8x - 2x^2$$

sketch $y=4x-x^2$
quadratic, opens down,
zeros $x=0,4$. vertex $(2,4)$



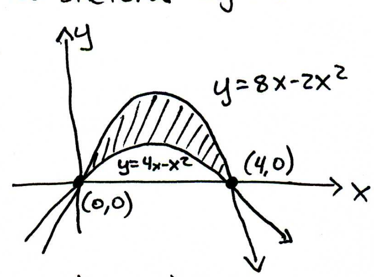
Test point $(2,0)$:
$0 \ge 4(2)-(2)^2$
$0 \ge 4$ False.
shade side doesn't
contain $(2,0)$.

sketch $y=8x-2x^2$
quadratic, opens down,
zeros $x=0,4$ vertex $(2,8)$



Test Point $(2,0)$
$0 \le 8(2)-2(2)^2$
$0 \le 8$ True.
shade side containing $(2,0)$.

Put sketches together:



Describe region:
for $x \in [0,4]$ y varies
from $y=4x-x^2$ to
$y=8x-2x^2$.