When adding or subtracting rational expressions you might have to do a lot of work. In general, you might need to

- factor any polynomials in the expressions
- get a common denominator for the rational expressions **(the critical step!)**
- add or subtract using $\frac{a}{c} \pm \frac{b}{c} = \frac{a \pm b}{c}$
- simplify the numerator (this could even involve another factoring!)
- simplify further by canceling any common terms in the numerator and denominator

Be careful, show all your work, and make sure minus signs get distributed correctly; for example, $-3x(x+4)$ is equal to $-3x^2 - 12x$ NOT $-3x^2 + 12x$.

Questions

1. Simplify $\dfrac{x^2+3x-10}{x^2+x-20} \cdot \dfrac{x^2-3x-4}{x^2+4x+3}$.

2. Simplify $\dfrac{x^2-x-20}{x^2-3x-10} \cdot \dfrac{x^2+7x+10}{x^2+4x-5}$.

3. Simplify $(6x-5) \div \dfrac{36x^2-25}{6x^2+17x+10}$.

4. Simplify $\dfrac{4x^2-9}{4x^2+12x+9} \div (6x-9)$.

5. Simplify $\dfrac{3x^2+12xy+12y^2}{x^2+4xy+3y^2} \div \dfrac{4x+8y}{x+y}$.

6. Simplify $\dfrac{5y^2+17y+6}{10y^2+9y+2} \cdot \dfrac{4y^2-1}{2y^2+5y-3}$.

7. Simplify $\dfrac{x^2+8x+15}{2x^2+11x+5} \div \dfrac{x^2+6x+9}{2x^2-7x-4}$.

8. Simplify $\dfrac{8x+3}{5x+7} - \dfrac{6x+10}{5x+7}$.

9. Find the lowest common denominator for $\dfrac{1}{x^2-9}$ and $\dfrac{1}{x+3}$.

10. Find the lowest common denominator for $\dfrac{1}{2x^2-9x-35}$ and $\dfrac{1}{4x^2+20x+25}$.

11. Simplify $\dfrac{8}{cd} + \dfrac{9}{d}$.

12. Simplify $\dfrac{2}{y-1} + \dfrac{2}{y+1}$.

13. Simplify $\dfrac{2}{3xy} + \dfrac{1}{6yz}$.

14. Simplify $\dfrac{6}{3x-4} - \dfrac{5}{4x-3}$.

15. Simplify $\dfrac{x}{x^2+2x-3} - \dfrac{x}{x^2-5x+4}$.

16. Simplify $\dfrac{3x+5}{x^2+4x+3} + \dfrac{-x+5}{x^2+2x-3}$.

17. Simplify $\dfrac{2x}{x^2+5x+6} - \dfrac{x+1}{x^2+2x-3}$.

Solutions

1. Simplify $\dfrac{x^2+3x-10}{x^2+x-20}\cdot\dfrac{x^2-3x-4}{x^2+4x+3}$. Factor all polynomials:

$$\begin{aligned}
x^2+3x-10 &= (x-2)(x+5) \text{ two numbers whose product is } -10 \text{ sum is } 3\text{: } -2,5\\
x^2-3x-4 &= (x-4)(x+1) \text{ two numbers whose product is } -4 \text{ sum is } -3\text{: } -4,1\\
x^2+x-20 &= (x+5)(x-4) \text{ two numbers whose product is } -20 \text{ sum is } 1\text{: } -4,5\\
x^2+4x+3 &= (x+3)(x+1) \text{ two numbers whose product is } 3 \text{ sum is } 4\text{: } 1,3
\end{aligned}$$

$$\begin{aligned}
\frac{x^2+3x-10}{x^2+x-20}\cdot\frac{x^2-3x-4}{x^2+4x+3} &= \frac{(x^2+3x-10)(x^2-3x-4)}{(x^2+x-20)(x^2+4x+3)} \text{ Simplify polynomial multiplication.}\\
&= \frac{(x-2)\cancel{(x+5)}\cancel{(x-4)}\cancel{(x+1)}}{\cancel{(x+5)}\cancel{(x-4)}(x+3)\cancel{(x+1)}}\\
&= \frac{x-2}{x+3} \text{ and } x-4\neq 0, x+1\neq 0, x+5\neq 0
\end{aligned}$$

2. Simplify $\dfrac{x^2-x-20}{x^2-3x-10}\cdot\dfrac{x^2+7x+10}{x^2+4x-5}$. Factor all polynomials:

$$\begin{aligned}
x^2-x-20 &= (x-5)(x+4) \text{ two numbers whose product is } -20 \text{ sum is } -1\text{: } -5,4\\
x^2-3x-10 &= (x-5)(x+2) \text{ two numbers whose product is } -10 \text{ sum is } -3\text{: } -5,2\\
x^2+7x+10 &= (x+5)(x+2) \text{ two numbers whose product is } 10 \text{ sum is } 7\text{: } 5,2\\
x^2+4x-5 &= (x+5)(x-1) \text{ two numbers whose product is } -5 \text{ sum is } 4\text{: } 5,-1
\end{aligned}$$

$$\begin{aligned}
\frac{x^2-x-20}{x^2-3x-10}\cdot\frac{x^2+7x+10}{x^2+4x-5} &= \frac{(x^2-x-20)(x^2-3x-10)}{(x^2+7x+10)(x^2+4x-5)} \text{ Simplify polynomial multiplication.}\\
&= \frac{\cancel{(x-5)}(x+4)\cancel{(x+5)}\cancel{(x+2)}}{\cancel{(x-5)}\cancel{(x+2)}\cancel{(x+5)}(x-1)}\\
&= \frac{x+4}{x-1} \text{ and } x+5\neq 0, x-5\neq 0, x+2\neq 0
\end{aligned}$$

3. Simplify $(6x-5)\div\dfrac{36x^2-25}{6x^2+17x+10}$. Factor all polynomials:

$$\begin{aligned}
6x^2+17x+10 &= 6x^2+12x+5x+10 \text{ two numbers whose product is } 60 \text{ sum is } 17\text{: } 12,5\\
&= 6x(x+2)+5(x+2) \text{ Factor by grouping}\\
&= (6x+5)(x+2)\\
36x^2-25 &= (6x-5)(6x+5) \text{ Difference of squares}
\end{aligned}$$

$$\begin{aligned}
(6x-5)\div\frac{36x^2-25}{6x^2+17x+10} &= (6x-5)\cdot\frac{6x^2+17x+10}{36x^2-25} \text{ Simplify polynomial division.}\\
&= \frac{(6x-5)(6x^2+17x+10)}{(36x^2-25)} \text{ Simplify polynomial multiplication.}\\
&= \frac{\cancel{(6x-5)}\cancel{(6x+5)}(x+2)}{\cancel{(6x-5)}\cancel{(6x+5)}}\\
&= x+2 \text{ and } 6x+5\neq 0, 6x-5\neq 0
\end{aligned}$$

4. Simplify $\dfrac{4x^2-9}{4x^2+12x+9} \div (6x-9)$. Factor all polynomials:

$$\begin{aligned}
4x^2+12x+9 &= 4x^2+6x+6x+9 \text{ two numbers whose product is 36 sum is 12: } 6,6\\
&= 2x(2x+3)+3(2x+3) \text{ Factor by grouping}\\
&= (2x+3)(2x+3) \text{ hey–this was a perfect square!}\\
4x^2-9 &= (2x+3)(2x-3) \text{ Difference of squares}\\
6x-9 &= 3(2x-3) \text{ common factor}
\end{aligned}$$

$$\begin{aligned}
\frac{4x^2-9}{4x^2+12x+9} \div (6x-9) &= \frac{4x^2-9}{4x^2+12x+9} \cdot \frac{1}{(6x-9)} \text{ Simplify polynomial division.}\\
&= \frac{(4x^2-9)}{(4x^2+12x+9)(6x-9)} \text{ Simplify polynomial multiplication.}\\
&= \frac{\cancel{(2x-3)}\cancel{(2x+3)}}{(2x+3)\cancel{(2x+3)}3\cancel{(2x-3)}}\\
&= \frac{1}{3(2x+3)} \text{ and } 2x-3\neq 0, 2x+3\neq 0
\end{aligned}$$

5. Simplify $\dfrac{3x^2+12xy+12y^2}{x^2+4xy+3y^2} \div \dfrac{4x+8y}{x+y}$. Factor all polynomials (let the y tag along with the constants):

$$\begin{aligned}
3x^2+12xy+12y^2 &= 3x^2+6xy+6xy+12y^2 \text{ two numbers whose product is 36 sum is 12: } 6,6\\
&= 3x(x+2y)+6y(x+2y) \text{ Factor by grouping}\\
&= (3x+6y)(x+2y) = 3(x+2y)(x+2y) \text{ hey–this was a perfect square!}\\
x^2+4xy+3y^2 &= (x+1y)(x+3y) = (x+y)(x+3y) \text{ two numbers whose product is 3 sum is 4: } 1,3\\
4x+8y &= 4(x+2y) \text{ common factor}
\end{aligned}$$

$$\begin{aligned}
\frac{3x^2+12xy+12y^2}{x^2+4xy+3y^2} \div \frac{4x+8y}{x+y} &= \frac{3x^2+12xy+12y^2}{x^2+4xy+3y^2} \cdot \frac{x+y}{4x+8y} \text{ Simplify polynomial division.}\\
&= \frac{(3x^2+12xy+12y^2)(x+y)}{(x^2+4xy+3y^2)(4x+8y)} \text{ Simplify polynomial multiplication.}\\
&= \frac{3(x+2y)\cancel{(x+2y)}\cancel{(x+y)}}{\cancel{(x+y)}(x+3y)4\cancel{(x+2y)}}\\
&= \frac{3(x+2y)}{4(x+3y)} \text{ and } x+2y\neq 0, x+y\neq 0
\end{aligned}$$

6. Simplify $\dfrac{5y^2+17y+6}{10y^2+9y+2}\cdot\dfrac{4y^2-1}{2y^2+5y-3}$. Factor all polynomials:

$$\begin{aligned}
5y^2+17y+6 &= 5y^2+15y+2y+6 \text{ two numbers whose product is 30 sum is 17: } 15, 2\\
&= 5y(y+3)+2(y+3) \text{ Factor by grouping}\\
&= (5y+2)(y+3)\\
10y^2+9y+2 &= 10y^2+5y+4y+2 \text{ two numbers whose product is 20 sum is 9: } 5, 4\\
&= 5y(2y+1)+2(2y+1) \text{ Factor by grouping}\\
&= (5y+2)(2y+1)\\
2y^2+5y-3 &= 2y^2+6y-1y-3 \text{ two numbers whose product is } -6 \text{ sum is 5: } 6, -1\\
&= 2y(y+3)-1(y+3) \text{ Factor by grouping}\\
&= (2y-1)(y+3)\\
4y^2-1 &= (2y-1)(2y+1) \text{ difference of squares}
\end{aligned}$$

$$\begin{aligned}
\frac{5y^2+17y+6}{10y^2+9y+2}\cdot\frac{4y^2-1}{2y^2+5y-3} &= \frac{(5y^2+17y+6)(4y^2-1)}{(10y^2+9y+2)(2y^2+5y-3)} \text{ Simplify polynomial multiplication.}\\
&= \frac{\cancel{(5y+2)}\cancel{(y+3)}\cancel{(2y+1)}\cancel{(2y-1)}}{\cancel{(5y+2)}\cancel{(2y+1)}\cancel{(2y-1)}\cancel{(y+3)}}\\
&= 1 \text{ and } 5y+2\neq 0, y+3\neq 0, 2y+1\neq 0, 2y-1\neq 0
\end{aligned}$$

7. Simplify $\dfrac{x^2+8x+15}{2x^2+11x+5}\div\dfrac{x^2+6x+9}{2x^2-7x-4}$. Factor all polynomials:

$$\begin{aligned}
x^2+8x+15 &= (x+5)(x+3) \text{ two numbers whose product is 15 sum is 8: } 5, 3\\
2x^2+11x+5 &= 2x^2+10x+1x+5 \text{ two numbers whose product is 10 sum is 11: } 10, 1\\
&= 2x(x+5)+1(x+5) \text{ Factor by grouping}\\
&= (2x+1)(x+5)\\
2x^2-7x-4 &= 2x^2-8x+1x-4 \text{ two numbers whose product is } -8 \text{ sum is } -7: -8, 1\\
&= 2x(x-4)+1(x-4) \text{ Factor by grouping}\\
&= (2x+1)(x-4)\\
x^2+6x+9 &= (x+3)(x+3) \text{ two numbers whose product is 9 sum is 6: } 3, 3
\end{aligned}$$

$$\begin{aligned}
\frac{x^2+8x+15}{2x^2+11x+5}\div\frac{x^2+6x+9}{2x^2-7x-4} &= \frac{x^2+8x+15}{2x^2+11x+5}\cdot\frac{2x^2-7x-4}{x^2+6x+9} \text{ Simplify polynomial division.}\\
&= \frac{(x^2+8x+15)(2x^2-7x-4)}{(2x^2+11x+5)(x^2+6x+9)}\cdot\frac{2x^2-7x-4}{x^2+6x+9} \text{ Simplify polynomial multiplication.}\\
&= \frac{\cancel{(x+5)}\cancel{(x+3)}\cancel{(2x+1)}(x-4)}{\cancel{(2x+1)}\cancel{(x+5)}\cancel{(x+3)}(x+3)}\\
&= \frac{x-4}{x+3} \text{ and } x+5\neq 0, x+3\neq 0, 2x+1\neq 0
\end{aligned}$$

8. The denominators are the same, so we can subtract immediately.

$$\begin{aligned}
\frac{8x+3}{5x+7}-\frac{6x+10}{5x+7} &= \frac{(8x+3)-(6x+10)}{5x+7} \text{ subtract rational expressions with common denominators}\\
&= \frac{8x+3-6x-10}{5x+7}\\
&= \frac{2x-7}{5x+7}
\end{aligned}$$

9. To find lowest common denominator we need to factor.

$$x^2 - 9 = (x+3)(x-3) \text{ difference of squares}$$
$$x + 3 = (x+3)$$

The lowest common denominator is $(x+3)(x-3)$. I've highlighted the overlap in red.

10. Factor everything first.

$$\begin{aligned} 2x^2 - 9x - 35 &= 2x^2 - 14x + 5x - 35 \text{ need two numbers whose product is } -70 \text{ and sum is } -9\text{: } -14, 5 \\ &= 2x(x-7) + 5(x-7) \text{ factor by grouping} \\ &= (2x+5)(x-7) \\ 4x^2 + 20x + 25 &= 4x^2 + 10x + 10x + 25 \text{ need two numbers whose product is } 100 \text{ and sum is } 20\text{: } 10, 10 \\ &= 2x(2x+5) + 5(2x+5) \text{ factor by grouping} \\ &= (2x+5)(2x+5) \text{ this was a perfect square} \end{aligned}$$

$$2x^2 - 9x - 35 = (2x+5)(x-7)$$
$$4x^2 + 20x + 25 = (2x+5)(2x+5)$$

LCD is $(2x+5)(2x+5)(x-7)$

11. Nothing needs to be factored.

$$\begin{aligned} \frac{8}{cd} + \frac{9}{d} &= \frac{8}{cd} + \frac{9 \cdot c}{d \cdot c} \text{ multiply by appropriate quantities to make the denominators the same.} \\ &= \frac{8}{cd} + \frac{9c}{cd} \\ &= \frac{8+9c}{cd} \text{ add rational expressions with common denominators} \end{aligned}$$

12. Nothing needs to be factored.

$$\begin{aligned} \frac{2}{y-1} + \frac{2}{y+1} &= \frac{2(y+1)}{(y-1)(y+1)} + \frac{2(y-1)}{(y+1)(y-1)} \text{ multiply by appropriate quantities to make the denominators the same.} \\ &= \frac{2(y+1) + 2(y-1)}{(y-1)(y+1)} \text{ add rational expressions with common denominators} \\ &= \frac{2y + 2 + 2y - 2}{(y-1)(y+1)} \text{ simplify numerator} \\ &= \frac{4y}{(y-1)(y+1)} \end{aligned}$$

13. Nothing needs to be factored.

$$\begin{aligned} \frac{2}{3xy} + \frac{1}{6yz} &= \frac{2(2z)}{3xy(2z)} + \frac{1(x)}{6yz(x)} \\ &= \frac{4z}{6xyz} + \frac{x}{6xyz} \\ &= \frac{4z+x}{6xyz} \end{aligned}$$

14. Nothing needs to be factored.

$$\begin{aligned}\frac{6}{3x-4}-\frac{5}{4x-3} &= \frac{6(4x-3)}{(3x-4)(4x-3)}-\frac{5(3x-4)}{(4x-3)(3x-4)}\\ &= \frac{6(4x-3)-5(3x-4)}{(3x-4)(4x-3)}\\ &= \frac{24x-18-15x+20}{(3x-4)(4x-3)}\\ &= \frac{9x+2}{(3x-4)(4x-3)}\end{aligned}$$

15. We need to factor here.

$$\begin{aligned}x^2+2x-3 &= (x+3)(x-1) \text{ two numbers whose product is } -3 \text{ sum is } 2\text{: } 3,-1\\ x^2-5x+4 &= (x-4)(x-1) \text{ two numbers whose product is } 4 \text{ sum is } -5\text{: } -4,-1\end{aligned}$$

$$\begin{aligned}\frac{x}{x^2+2x-3}-\frac{x}{x^2-5x+4} &= \frac{x}{(x+3)(x-1)}-\frac{x}{(x-4)(x-1)}\\ &= \frac{x(x-4)}{(x+3)(x-1)(x-4)}-\frac{x(x+3)}{(x-4)(x-1)(x+3)} \text{ get common denominator}\\ &= \frac{x(x-4)-x(x+3)}{(x+3)(x-1)(x-4)} \text{ subtract now that we have common denominator}\\ &= \frac{x^2-4x-x^2-3x}{(x+3)(x-1)(x-4)} \text{ simplify}\\ &= \frac{-7x}{(x+3)(x-1)(x-4)}\end{aligned}$$

16. We need to factor here.

$$\begin{aligned}x^2+4x+3 &= (x+3)(x+1) \text{ two numbers whose product is } 3 \text{ sum is } 4\text{: } 3,1\\ x^2+2x-3 &= (x+3)(x-1) \text{ two numbers whose product is } -3 \text{ sum is } 2\text{: } 3,-1\end{aligned}$$

$$\begin{aligned}\frac{3x+5}{x^2+4x+3}+\frac{-x+5}{x^2+2x-3} &= \frac{3x+5}{(x+3)(x+1)}+\frac{5-x}{(x+3)(x-1)} \text{ factor}\\ &= \frac{(3x+5)(x-1)}{(x+3)(x+1)(x-1)}+\frac{(5-x)(x+1)}{(x+3)(x-1)(x+1)}\\ &= \frac{(3x+5)(x-1)}{(x+3)(x+1)(x-1)}+\frac{(5-x)(x+1)}{(x+3)(x-1)(x+1)} \text{ get common denominator}\\ &= \frac{(3x+5)(x-1)+(5-x)(x+1)}{(x+3)(x+1)(x-1)} \text{ add}\\ &= \frac{3x^2+2x-5-x^2+4x+5}{(x+3)(x+1)(x-1)} \text{ simplify numerator: distribute}\\ &= \frac{2x^2+6x}{(x+3)(x+1)(x-1)} \text{ simplify numerator: collect like terms}\\ &= \frac{2x\cancel{(x+3)}}{\cancel{(x+3)}(x+1)(x-1)} \text{ simplify: factor numerator}\\ &= \frac{2x}{(x+1)(x-1)} \text{ and } x+3\neq 0\end{aligned}$$

17. We need to factor here.

$$x^2+5x+6=(x+3)(x+2) \text{ two numbers whose product is 6 sum is 5: } 3,2$$
$$x^2+2x-3=(x+3)(x-1) \text{ two numbers whose product is } -3 \text{ sum is 2: } 3,-1$$

$$\begin{aligned}
\frac{2x}{x^2+5x+6}-\frac{x+1}{x^2+2x-3} &= \frac{2x}{(x+3)(x+2)}-\frac{x+1}{(x+3)(x-1)} \text{ factor}\\
&= \frac{2x(x-1)}{(x+3)(x+2)(x-1)}-\frac{(x+1)(x+2)}{(x+3)(x-1)(x+2)} \text{ get common denominator}\\
&= \frac{2x(x-1)-(x+1)(x+2)}{(x+3)(x+2)(x-1)} \text{ subtract}\\
&= \frac{2x^2-2x-x^2-3x-2}{(x+3)(x+2)(x-1)} \text{ simplify numerator: distribute}\\
&= \frac{x^2-5x-2}{(x+3)(x+2)(x-1)} \text{ simplify numerator: collect like terms}
\end{aligned}$$

The numerator is prime. If we could factor it, we would.