Note: You can expect other types of questions on the test than the ones presented here!

Questions

Example 1. Find the vertex of the quadratic $f(x) = 14x^2 - x$.

Example 2. Given $f(x) = x^3 + 1$, simplify the quantity $\dfrac{f(x+h) - f(x)}{h}$ as much as possible. You should simplify until substituting zero for h will not yield the indeterminant form $\dfrac{0}{0}$.

Example 3. Find the remainder $r(x)$ when $g(x) = -4x^3 - 2x + 3$ is divided by $d(x) = 2x - 8$ using long division of polynomials.

Example 4. Sketch the polynomial $f(x) = (2x-1)^3(2-x)^2$ by hand. Show all your work.

Example 5. For the function $g(x)$ given below, determine what monomial the function approaches for large x. Then, evaluate $\lim\limits_{x\to\infty} g(x)$ and $\lim\limits_{x\to-\infty} g(x)$. Does the function $g(x)$ have any horizontal asymptotes?

$$g(x) = \frac{(-x^4 + 24x - 78)(-2x + 1)}{3x^3 - 99}$$

Example 6. Sketch the rational function $h(x) = \dfrac{(x+6)^3}{2(x^2-4)}$ by hand
(find x-intercepts, vertical asymptotes, slant or horizontal asymptotes, and end behaviour).

Example 7. Solve the inequality $\dfrac{|x-2|(-4x-5)}{x-5} \le 0$ by constructing a sign chart, or drawing an appropriate sketch by hand. Show your work.

Example 8. Solve the inequality $\dfrac{1}{x+2} \le -\dfrac{1}{x^2}$ by constructing a sign chart, or drawing an appropriate sketch by hand. Show all your work.

Example 9. Solve $\dfrac{3x}{x+1} + \dfrac{5}{x-2} = \dfrac{15}{x^2 - x - 2}$ for x.

Example 10. The volume of an enclosed gas (at a constant temperature) varies inversely as the pressure. This means the relationship between volume V and pressure P can be written as

$$V = \frac{k}{P} \qquad \text{(Boyle's Law)}$$

where k is the proportionality constant.

If the pressure of a 3.46 L sample of neon gas at 302° K is 0.926 atm, what would the volume be at a pressure of 1.452 atm if the temperature does not change? Since this problem is from chemistry, you can have a solution that uses decimals rather than fractions.

Example 11. Show that $x = -2i$ is a root of $f(x) = 4x^3 - 7x^2 + 16x - 28$. Is $x = 2i$ an x-intercept of f?

Solutions

Example 1. Find the vertex of the quadratic $f(x) = 14x^2 - x$.

Solution using completing the square:

$$\begin{aligned}
f(x) &= 14x^2 - x \\
&= 14\left(x^2 + \left(\frac{-1}{14}\right)x\right) + 3 \\
&= 14\left(x^2 + \left(\frac{-1}{14}\right)x + \left(\frac{-1}{28}\right)^2 - \left(\frac{-1}{28}\right)^2\right) \\
&= 14\left(x^2 + \left(\frac{-1}{14}\right)x + \left(\frac{-1}{28}\right)^2 - \left(\frac{-1}{28}\right)^2\right) \\
&= 14\left(\left[x + \left(\frac{-1}{28}\right)\right]^2 - \left(\frac{-1}{28}\right)^2\right) \\
&= 14\left(\left[x - \frac{1}{28}\right]^2 - \left(\frac{-1}{28}\right)^2\right) \\
&= 14\left[x - \frac{1}{28}\right]^2 - 14\left(\frac{-1}{28}\right)^2 \\
&= 14\left[x - \frac{1}{28}\right]^2 - \frac{1}{56}
\end{aligned}$$

The vertex is $(h, k) = \left(\frac{1}{28}, -\frac{1}{56}\right)$.

Another way to solve this would be to recognize that the x coordinate of the vertex is always centered between the zeros (even if the zeros are not real numbers!).

Zeros:

$$\begin{aligned}
14x^2 - x &= 0 \\
x(14x - 1) &= 0 \\
x &= 0 \text{ or } x = 1/14
\end{aligned}$$

So the x coordinate of the vertex is $\frac{1}{2}\left(0 + \frac{1}{14}\right) = \frac{1}{28}$.

The y coordinate of the vertex is $f(1/28) = 14(1/28)^2 - (1/28) = -1/56$.

The vertex is $(h, k) = \left(\frac{1}{28}, -\frac{1}{56}\right)$.

Example 2. Given $f(x) = x^3 + 1$, simplify the quantity $\dfrac{f(x+h) - f(x)}{h}$ as much as possible. You should simplify until substituting zero for h will not yield the indeterminant form $\dfrac{0}{0}$.

$$
\begin{aligned}
\text{Average Rate of Change} &= \frac{f(x+h) - f(x)}{h} \\
&= \frac{(x+h)^3 + 1 - (x)^3 - 1}{h} \\
&= \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h} \\
&= \frac{3x^2h + 3xh^2 + h^3}{h} \\
&= \frac{h(3x^2 + 3xh + h^2)}{h} \\
&= 3x^2 + 3xh + h^2
\end{aligned}
$$

Example 3. Find the remainder $r(x)$ when $g(x) = -4x^3 - 2x + 3$ is divided by $d(x) = 2x - 8$ using long division of polynomials.

$$
\begin{array}{rl}
 & -2x^2 - 8x - 33 \\
2x - 8 \,\big) & -4x^3 - 0x^2 - 2x + 3 \\
 & -4x^3 + 16x^2 \qquad \text{subtract} \\
\hline
 & -16x^2 - 2x + 3 \\
 & -16x^2 + 64x \qquad \text{subtract} \\
\hline
 & -66x + 3 \\
 & -66x + 264 \qquad \text{subtract} \\
\hline
 & -261 \leftarrow \text{remainder.}
\end{array}
$$

Example 4. Sketch the polynomial $f(x) = (2x-1)^3(2-x)^2$ by hand. Show all your work.

Zeros:

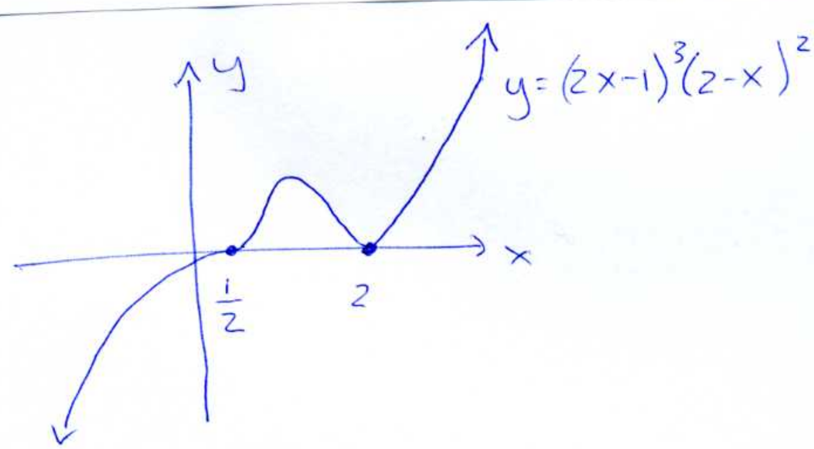
$x = \frac{1}{2}$, multiplicity 3 (odd), so f changes sign.

$x = 2$, multiplicity 2 (even), so f does not change sign.

End behaviour: for large x,

$$f(x) = (2x-1)^3(2-x)^2 \sim (2x)^3(-x)^2 = 8x^5.$$

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} 8x^5 = \infty \qquad \lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} 8x^5 = -\infty.$$



Example 5. For the function $g(x)$ given below, determine what monomial the function approaches for large x. Then, evaluate $\lim_{x\to\infty} g(x)$ and $\lim_{x\to-\infty} g(x)$. Does the function $g(x)$ have any horizontal asymptotes?

$$g(x) = \frac{(-x^4 + 24x - 78)(-2x+1)}{3x^3 - 99}$$

For end behaviour, we look at the leading terms in each factor, since the leading terms will dominate for large $|x|$:

$$\begin{aligned} g(x) &= \frac{(-x^4 + 24x - 78)(-2x+1)}{3x^3 - 99} \\ &\sim \frac{(-x^4)(-2x)}{3x^3} = \frac{2}{3}x^2 \text{ for } |x| \text{ large.} \end{aligned}$$

Therefore, the end behaviour can be described as: $\lim_{x\to\infty} f(x) = \infty \qquad \lim_{x\to-\infty} f(x) = \infty$.

The function does not have any horizontal asymptotes.

Example 6. Sketch the rational function $h(x) = \dfrac{(x+6)^3}{2(x^2-4)}$ by hand (find x-intercepts, vertical asymptotes, slant or horizontal asymptotes, and end behaviour).

The numerator is already factored.

The denominator factors as $2x^2 - 8 = 2(x^2-4) = 2(x+2)(x-2)$.

Therefore, $h(x) = \dfrac{(x+6)^3}{2x^2-8} = \dfrac{(x+6)^3}{2(x+2)(x-2)}$.

The zero of the numerator is $x = 6$, which is multiplicity 3 (odd), so the function will cross the x-axis here. Since the multiplicity is greater than 2, the graph will be flat (horizontal) near $x = 6$.

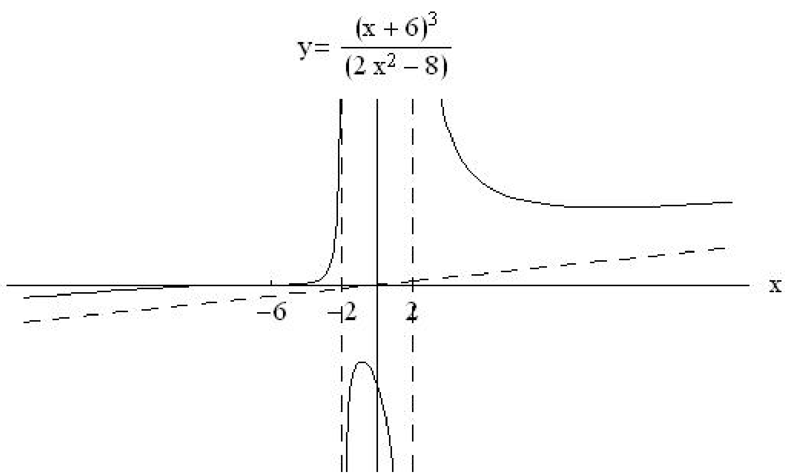
The denominator is zero when $x = 2$ and $x = -2$, so these are vertical asymptotes. Since the multiplicity of these points is odd, the function will change sign at $x = 2$ and $x = -2$.

For end behaviour, we can look at what happens for $|x|$ very large:

$$h(x) = \frac{(x+6)^3}{2x^2-8} \sim \frac{(x)^3}{2x^2} = \frac{x}{2}.$$

This means that for $|x|$ very large the function $h(x)$ will approach the straight line $y = x/2$. This is a slant asymptote.

We have enough information to plot the function. The dashed line in the plot is the slant asymptote $y = x/2$.



Example 7. Solve the inequality $\dfrac{|x-2|(-4x-5)}{x-5} \leq 0$ by constructing a sign chart, or drawing an appropriate sketch by hand. Show your work.

I will solve this using a sign chart, and examining the sign of the factors.

The numerator of the function $f(x) = \dfrac{|x-2|(-4x-5)}{x-5}$ is zero when $x = -5/4, x = 2$.

The denominator of f is zero when $x = 5$.

$\frac{\lvert - \rvert(+)}{(-)}$	0	$\frac{\lvert - \rvert(-)}{(-)}$	0	$\frac{\lvert + \rvert(-)}{(-)}$	∞	$\frac{\lvert + \rvert(-)}{(+)}$	
negative	$-\frac{5}{4}$	positive	2	positive	5	negative	x

From the sign chart, we see that $\dfrac{|x-2|(-4x-5)}{x-5} \leq 0$ if $x \in (-\infty, -5/4] \cup (2,2) \cup (5, \infty)$. We exclude $x = 5$, since the function is not defined there.

Example 8. Solve the inequality $\dfrac{1}{x+2} \leq -\dfrac{1}{x^2}$ by constructing a sign chart, or drawing an appropriate sketch by hand. Show all your work.

We need to write this as a single rational function, rather than as a sum of rational functions, before we can construct our sign chart or a sketch.

$$\begin{aligned}
\frac{1}{x+2} + \frac{1}{x^2} &\leq 0 \\
\frac{1}{x+2}\left(\frac{x^2}{x^2}\right) + \frac{1}{x^2}\left(\frac{x+2}{x+2}\right) &\leq 0 \\
\frac{x^2+x+2}{(x+2)(x^2)} &\leq 0
\end{aligned}$$

Let's construct a sign chart.

The quadratic in the numerator has no real roots.

The denominator is zero if $x = -2, 0$. These are the possible values where the function will change sign.

$\frac{(+)}{(-)(+)}$	∞	$\frac{(+)}{(+)(+)}$	∞	$\frac{(+)}{(+)(+)}$	x
negative	-2	positive	0	positive	

From the sign chart, we see that $\dfrac{1}{x+2} \leq -\dfrac{1}{x^2}$ if $x \in (-\infty, -2)$.

Example 9. Solve $\dfrac{3x}{x+1} + \dfrac{5}{x-2} = \dfrac{15}{x^2-x-2}$ for x.

$$\begin{aligned}
\frac{3x}{x+1} + \frac{5}{x-2} &= \frac{15}{x^2-x-2} \\
\frac{3x}{x+1} + \frac{5}{x-2} &= \frac{15}{(x-2)(x+1)} \\
(x-2)(x+1)\left[\frac{3x}{x+1} + \frac{5}{x-2} = \frac{15}{(x-2)(x+1)}\right] & \qquad \text{multiply by LCD } (x-2)(x+1) \\
3x(x-2) + 5(x+1) = 15 & \qquad x \neq 2, -1 \\
3x^2 - 6x + 5x + 5 - 15 = 0 & \qquad x \neq 2, -1 \\
3x^2 - x - 10 = 0 & \qquad x \neq 2, -1 \\
(3x+5)(x-2) = 0 & \qquad x \neq 2, -1
\end{aligned}$$

The solution to the original rational equation are $x = -5/3$. The $x = 2$ is an extraneous solution.

Example 10. The volume of an enclosed gas (at a constant temperature) varies inversely as the pressure. This means the relationship between volume V and pressure P can be written as

$$V = \frac{k}{P} \qquad \text{(Boyle's Law)}$$

where k is the proportionality constant.

If the pressure of a 3.46 L sample of neon gas at 302° K is 0.926 atm, what would the volume be at a pressure of 1.452 atm if the temperature does not change? Since this problem is from chemistry, you can have a solution that uses decimals rather than fractions.

The relation we have is

$$V = \frac{k}{P}$$

We can use the first data point to determine the proportionality constant k:

$$k = VP = (3.46)(0.926) = 3.20396.$$

The relationship is therefore

$$V = \frac{3.20396}{P}$$

and we have

$$V = \frac{3.20396}{1.452} = 2.20658.$$

So the volume would be 2.20658 L.

Example 11. Show that $x = -2i$ is a root of $f(x) = 4x^3 - 7x^2 + 16x - 28$. Is $x = 2i$ an x-intercept of f?

$$\begin{aligned}
f(x) &= 4x^3 - 7x^2 + 16x - 28 \\
f(2i) &= 4(2i)^3 - 7(2i)^2 + 16(2i) - 28 \\
&= 4(2i)^3 - 7(2i)^2 + 16(2i) - 28 \\
&= 4(2^3 i^3) - 7(2^2 i^2) + 32i - 28 \\
&= 32i^2 \cdot i - 28(-1) + 32i - 28 \\
&= -32i + 28 + 32i - 28 \\
&= 0
\end{aligned}$$

$x = 2i$ is not an x-intercept since x-intercepts must be real numbers.