Note: You can expect other types of questions on the test than the ones presented here!

Questions

Example 1. Given $f(x) = -3\ln(2x) + \ln(x^5)$, what is the domain of f? What is the range of f^{-1}? Find the algebraic expression for the inverse function $f^{-1}(x)$.

Example 2. What is the domain of $f(x) = \ln[x(x-1)]$?

Example 3. Sketch by hand the function $g(t) = e^{-3t} + 2$.

Example 4. The half life of a certain radioactive material is 65 days. There are initially 4 grams of the material present. Find an expression for the amount of material t **days** after the initial measurement of 4 grams. How many days will it take for there to be only 1 gram left? (give the answer both exactly and as a decimal).

Show all you calculations. Derive any formulas you need, do not simply plug numbers into a population growth formula you have memorized.

Example 5. Solve the equation $\dfrac{2^x - 2^{-x}}{3} = 4$ algebraically for x.

Example 6. Solve the equation $\dfrac{1}{2}\ln(x+3) = \ln x$.

Example 7. Solve the equation $\ln(t-2) + \ln(t+5) = 2\ln 3$.

Example 8. Given $f(x) = \ln(\sqrt{x})$, $g(x) = e^{x/4}$, and $h(x) = x^2$. Find the composition $(f \circ g \circ h)(x)$ and simplify as much as possible. Your final answer should **not** have exponentials and logarithms in it.

Example 9. A population of 200 fish is released into a lake. The population doubling time for this breed of fish is 15 months. If the plan is to allow limited fishing on the lake once the fish population exceeds 10000 fish, when should fishing be allowed to begin (give the answer both exactly and as a decimal)?

Show all you calculations. Derive any formulas you need, do not simply plug numbers into a population growth formula you have memorized.

Example 10. Find the inverse function $f^{-1}(x)$ if $f(x) = e^{-3x} + 2$. Verify you have the correct answer by checking that $f(f^{-1}(x)) = x$.

Solutions

Ex 1 Domain of f: $2x>0$ and $x^5>0$ $\Rightarrow$ $x>0$ is Domain of f.

Range of f^{-1} is the domain of f, so range of f^{-1} is $y>0$.

Inverse: $y=-3\ln(2x)+\ln(x^5)$

interchange x & y: $x=-3\ln(2y)+\ln(y^5)$

solve for y:

$$x=\ln\left((2y)^{-3}\right)+\ln(y^5)$$

$$x=\ln\left((2y)^{-3}y^5\right)$$

$$x=\ln\left(\frac{y^5}{8y^3}\right)$$

$$x=\ln\left(y^2/8\right)$$

$$e^x=y^2/8$$

$$y=\pm\sqrt{8}\,e^{x/2}$$

We have to pick one of these. Since range of f^{-1} is $y>0$, we have

$$y=f^{-1}(x)=\sqrt{8}\,e^{x/2}.$$

Ex 2 We need $x(x-1)>0$. Let's use a sign chart.

Zeros: $\left.\begin{array}{l}x=0\\x=1\end{array}\right\}$ multiplicity 1, odd, so there is a sign change at both zeros.

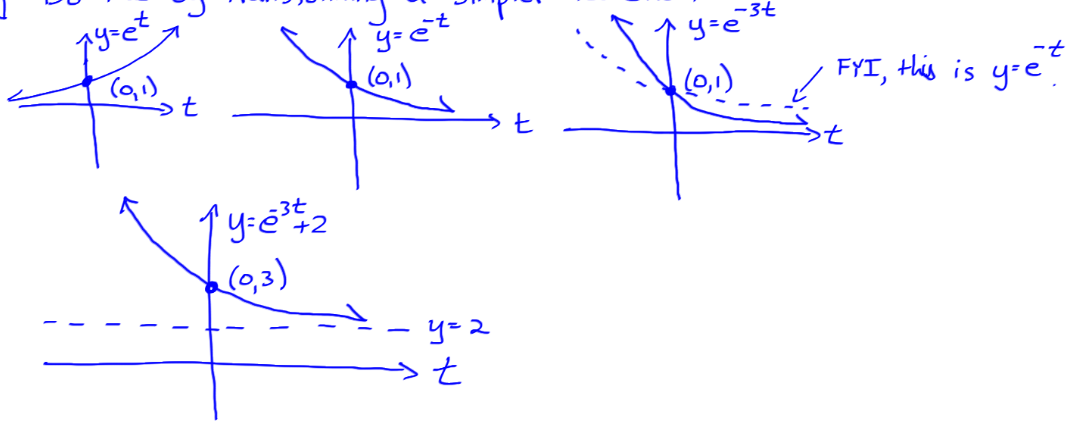
End behaviour: As $x\to\infty$, $x(x-1)\to x(x)=x^2>0$.

Sign chart for $x(x-1)$:

+	0	−	0	+
	0		1	

Domain is $x\in(-\infty,0)\cup(1,\infty)$.

Ex 3 Do this by transforming a simpler function.



Ex 4) Let t be in days.

$P(0) = 4$

$P(65) = \frac{1}{2} \cdot 4$

$P(2 \cdot 65) = \cancel{P(130)} = \left(\frac{1}{2}\right)^2 4$

$P(3 \cdot 65) = \cancel{P(195)} = \left(\frac{1}{2}\right)^3 \cdot 4$

$P(4 \cdot 65) = \left(\frac{1}{2}\right)^4 4$

Note we don't use these to find pattern

$\vdots$ look for pattern

$P(t) = \left(\frac{1}{2}\right)^{t/65} \cdot 4$

$= 2^{-t/65} \cdot 4$

either is fine.

There is 1 g left when

$1 = \left(\frac{1}{2}\right)^{t/65} \cdot 4$ solve for t.

$\frac{1}{4} = \left(\frac{1}{2}\right)^{t/65}$

$\ln(1/4) = \ln\left[\left(\frac{1}{2}\right)^{t/65}\right]$

$\ln(1/4) = \frac{t}{65} \ln\left(\frac{1}{2}\right)$

$t = \frac{65 \ln(1/4)}{\ln(1/2)}$ lots of other expressions are correct as well.

$t = 130$ days to have 1g left. (you can actually get this without using a calculator).

$t = 65 \frac{\ln\left(\left(\frac{1}{2}\right)^2\right)}{\ln\left(\frac{1}{2}\right)} = 65 \cdot 2 \frac{\ln(1/2)}{\ln(1/2)} = 130!$

Ex 5) $\left(2^x - 2^{-x} = 12\right) 2^x$ (it's going to be quadratic in $\omega = 2^x$)

$2^{2x} - 1 = 12 \cdot 2^x$

$(2^x)^2 - 12(2^x) - 1 = 0$ let $\omega = 2^x$

$\omega^2 - 12\omega - 1 = 0$ use quadratic formula.

$\omega = \frac{12 \pm \sqrt{144 + 4}}{2}$

$= 6 \pm \frac{\sqrt{148}}{2} = 6 \pm \sqrt{37}$

$\omega = 2^x = 6 + \sqrt{37}$ solve for x.

$\omega = 2^x = 6 - \sqrt{37}$ has no real solution, since $6 - \sqrt{37} < 0$ but $2^x > 0$.

$\ln(2^x) = \ln(6 + \sqrt{37})$

$x \ln(2) = \ln(6 + \sqrt{37})$

$x = \frac{\ln(6 + \sqrt{37})}{\ln(2)} \longleftarrow$ only solution to original equation.

Ex 6) $\ln\left((x+3)^{1/2}\right) = \ln x$

$$e^{\ln(\sqrt{x+3})} = e^{\ln x}$$

$$\sqrt{x+3} = x$$

$$x+3 = x^2$$

$$x^2 - x - 3 = 0$$

$$x = \frac{1 \pm \sqrt{1+12}}{2}$$

$$= \frac{1 \pm \sqrt{13}}{2}$$

In original equation, we required

$x+3>0$ and $x>0$ $\Rightarrow x>0$.

So $x = \frac{1-\sqrt{13}}{2}$ is an extraneous solution as it is less than zero.

Solution: $x = \frac{1+\sqrt{13}}{2}$.

Ex 7) $\ln\left[(t-2)(t+5)\right] = \ln 9$ (since $2\ln 3 = \ln(3^2) = \ln 9$)

$$e^{\ln[t^2+3t-10]} = e^{\ln 9}$$

$$t^2+3t-10 = 9$$

$$t^2+3t-19 = 0$$

$$t = \frac{-3 \pm \sqrt{9+4(19)}}{2}$$

$$t = \frac{-3 \pm \sqrt{85}}{2}$$

Domain of original equation is

$t-2>0$ and $t+5>0$ $\Rightarrow t>2$.

So exclude $t = \frac{-3-\sqrt{85}}{2}$ since it is less than 2.

$t = \frac{-3+\sqrt{85}}{2} > 2$ is the only solution.

Ex 8) $(f \circ g \circ h)(x) = f\left(g(h(x))\right)$

$$= f\left(g(x^2)\right)$$

$$= f\left(e^{x^2/4}\right)$$

$$= \ln\left[\sqrt{e^{x^2/4}}\right]$$

$$= \ln\left[\left(e^{x^2/4}\right)^{1/2}\right]$$

$$= \frac{1}{2}\ln\left[e^{x^2/4}\right]$$

$$= \frac{1}{2}\left(x^2/4\right) = \frac{x^2}{8}$$

Ex 9) Let t be in months.

$P(0) = 200$ fish.

$P(15) = 2 \cdot 200$

$P(2 \cdot 15) = 2^2 \cdot 200$

$P(3 \cdot 15) = 2^3 \cdot 200$

$P(4 \cdot 15) = 2^4 \cdot 200$

$\vdots$ look for pattern.

$P(t) = 2^{t/15} \cdot 200$

There will be 10000 fish when

$10\,000 = 2^{t/15} \cdot 200$ solve for t.

$50 = 2^{t/15}$

$\ln(50) = \ln\left(2^{t/15}\right)$

$\ln(50) = \frac{t}{15} \ln 2$

$t = \dfrac{15 \ln(50)}{\ln 2}$

~ 84.6

Fishing may begin 84.6 months after the fish are released.

Ex 10) $y = e^{-3x} + 2$

interchange x & y: $x = e^{-3y} + 2$

solve for y: $x - 2 = e^{-3y}$

$\ln(x-2) = -3y$

$y = -\frac{1}{3} \ln(x-2)$.

$f^{-1}(x) = -\frac{1}{3} \ln(x-2)$.

check:

$f(f^{-1}(x)) = f\left(-\frac{1}{3} \ln(x-2)\right)$

$= e^{-3\left(-\frac{1}{3} \ln(x-2)\right)} + 2$

$= e^{\ln(x-2)} + 2$

$= x - 2 + 2$

$= x$ correct!