Note: You can expect other types of questions on the test than the ones presented here!

Questions

Example 1 Eliminate the parameter and sketch the parametric curve given by $x = t - 2$, $y = 3/t$, $|t| < 1$.

Example 2 Solve the system of equations $x^2 + y^2 = 9$ and $x^2 - y = 2$. Sketch the situation to check that you found the correct number of solutions.

Example 3 Sketch the region defined by $x^2 - 2x + y^2 \geq 8$ and $y > |x + 1|$.

Example 4 A parabola is the set of all points in a plane equidistant from a particular line (the directrix) and a particular point (the focus) in the plane. If the focus is $(0, p)$ and the directrix is $y = -p$, derive the equation of the parabola $x^2 = 4yp$. Include a well labeled sketch of the situation to motivate your calculations.

Example 5 Sketch $4x^2 + y^2 - 32x + 16y + 124 = 0$. Locate the center, vertices, and foci of the ellipse.

Example 6 Sketch $9x^2 - 54x - 4y^2 + 8y + 113 = 0$. Locate the vertices and foci of the hyperbola.

Solutions

1) $x = t - 2 \longrightarrow t = x + 2$

substitute into $y = \frac{3}{t}$: $y = \frac{3}{x+2}$

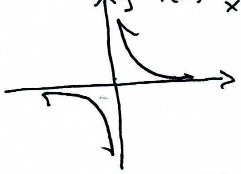
Domain: $|t| < 1 \longrightarrow -1 < t < 1$ At $t = -1$, $x = -1 - 2 = -3$

At $t = 1$, $x = +1 - 2 = -1$.

So parametric equation becomes

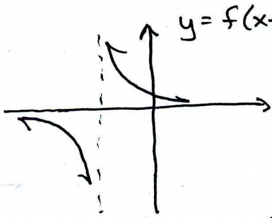
$$y = \frac{3}{x+2} \qquad -3 < x < -1.$$

$y = f(x) = \frac{1}{x}$



inside function $\rightarrow$ horizontal shift of 2 units to left.

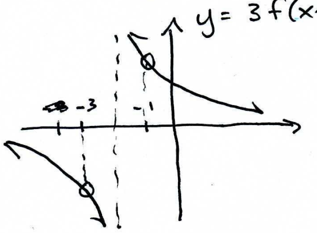
$y = f(x+2) = \frac{1}{x+2}$



$x = -2$

outside function $\rightarrow$ vertical stretch by factor of 3.

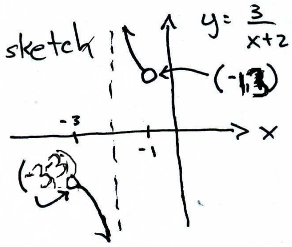
$y = 3f(x+2) = \frac{3}{x+2}$



$x = -2$

final sketch

$y = \frac{3}{x+2}$, $-3 < x < -1$



$x = -2$

2) $x^2+y^2=9$
$x^2-y=2 \longrightarrow y=x^2-2$ substitute into 1st equation.

$$x^2+(x^2-2)^2=9$$
$$x^2+x^4-4x^2+4=9$$
$$x^4-3x^2-5=0$$ difficult to factor. Backup and try something else.

$$\begin{array}{r} x^2+y^2=9 \\ \underline{x^2-y=2} \end{array} \quad \text{subtract}$$

$$y^2+y=7 \longrightarrow y^2+y-7=0$$

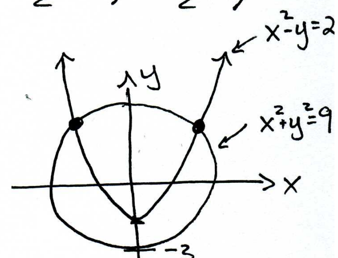
$$y=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$
$$=\frac{-1\pm\sqrt{1-4(1)(-7)}}{2}$$
$$=\frac{-1\pm\sqrt{29}}{2}$$

If $y=\frac{-1+\sqrt{29}}{2}$: $x=\pm\sqrt{2+y}$
$$=\pm\sqrt{2+\frac{-1+\sqrt{29}}{2}}$$
$$=\pm\sqrt{\frac{3+\sqrt{29}}{2}}$$

Solutions

$$\left(\sqrt{\frac{3+\sqrt{29}}{2}}, \frac{-1+\sqrt{29}}{2}\right)$$
$$\left(-\sqrt{\frac{3+\sqrt{29}}{2}}, \frac{-1+\sqrt{29}}{2}\right)$$

If $y=\frac{-1-\sqrt{29}}{2}$: $x=\pm\sqrt{2+y}$
$$=\pm\sqrt{\frac{3-\sqrt{29}}{2}}$$
since $\frac{3-\sqrt{29}}{2}<0$, no real solutions.



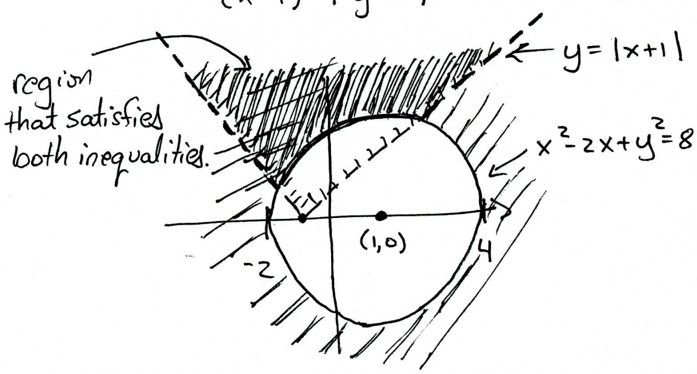
sketch shows there should only be 2 solutions.

3| Complete square:

$x^2-2x+y^2=8$

$\underbrace{x^2-2x+1}-1+y^2=8$

$(x-1)^2+y^2=9$ circle radius 3 center $(1,0)$.



Test point: $(1,0)$: $1^2-2(1)+0^2 \geq 8$

$-1 \geq 8$ False.

region we want is <u>outside</u> the circle.

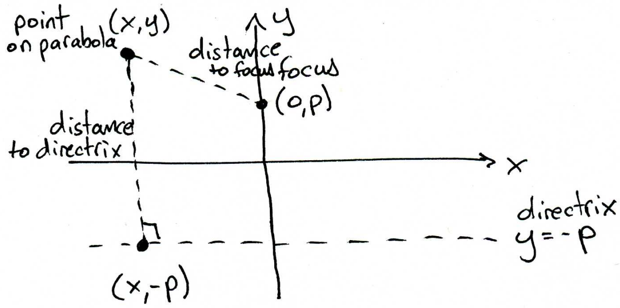
<u>$y > |x+1|$:</u> sketch $y=|x+1|$

which is $y=|x|$ shifted left 1 unit

Test point $(0,0)$: $0 > |1|$ False.

region we want is above $y=|x+1|$.

4)



distance to directrix = distance to focus

$$\sqrt{(x-x)^2+(y-(-p))^2} = \sqrt{(x-0)^2+(y-p)^2}$$

$$(y+p)^2 = x^2 + (y-p)^2$$

$$\cancel{y^2} + 2yp + \cancel{p^2} = x^2 + \cancel{y^2} - 2yp + \cancel{p^2}$$

$$2yp = x^2 - 2yp$$

$$x^2 = 4yp.$$

5| $4x^2 + y^2 - 32x + 16y + 124 = 0$

Complete square in x and y

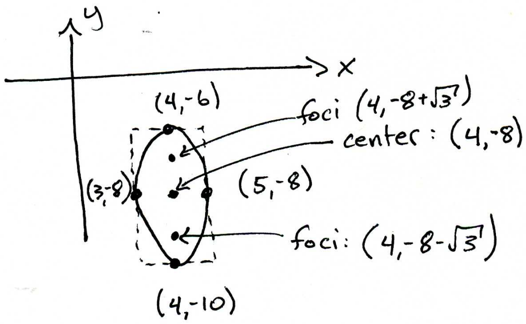
$$4[\underbrace{x^2 - 8x + 16}\ -16] + [\underbrace{y^2 + 16y + 64}-64] = -124$$

$$4[(x-4)^2 - 16] + [(y+8)^2 - 64] = -124$$

$$4(x-4)^2 - 64 + (y+8)^2 - 64 = -124$$

$$4(x-4)^2 + (y+8)^2 = 4$$

$$(x-4)^2 + \frac{(y+8)^2}{2^2} = 1 \quad \text{ellipse!}$$



Get Box: if $x=4$, then

$$\frac{(y+8)^2}{2^2} = 1$$

$$y+8 = \pm 2$$

$$y = -6, -10.$$

points $(4,-6)$ and $(4,-10)$

if $y=-8$, then

$$(x-4)^2 = 1$$

$$x-4 = \pm 1$$

$$x = 5, 3$$

points $(5,-8)$ and $(3,-8)$

Center $(4,-8)$

Vertices $(4,-6)$ and $(4,-10)$

Pythagorean relation: $c = \sqrt{2^2 - 1^2} = \sqrt{3}$

foci: $(4, -8+\sqrt{3})$ and $(4, -8-\sqrt{3})$

6) $9x^2 - 54x - 4y^2 + 8y + 113 = 0$

Complete square in both x and y:

$$9[\underbrace{x^2 - 6x + 9} - 9] - 4[\underbrace{y^2 - 2y + 1} - 1] = -113$$

$$9[(x-3)^2 - 9] - 4[(y-1)^2 - 1] = -113$$

$$9(x-3)^2 - 81 - 4(y-1)^2 + 4 = -113$$

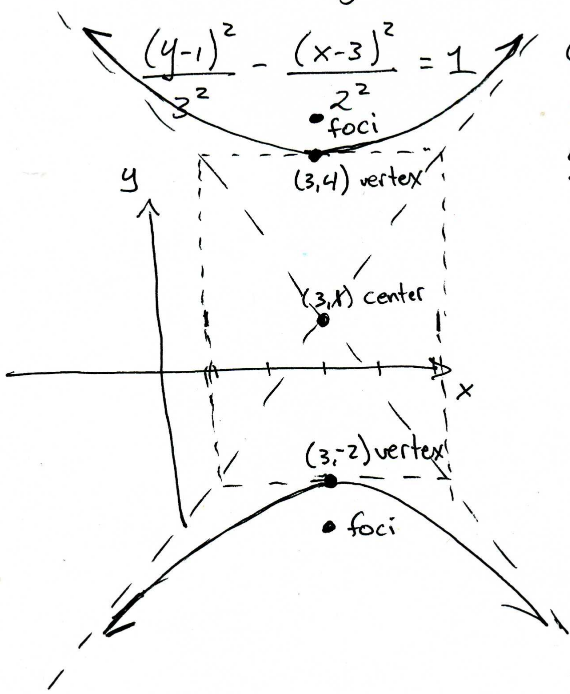
$$9(x-3)^2 - 4(y-1)^2 = -36$$

this should be positive
Multiply by -1.

$$4(y-1)^2 - 9(x-3)^2 = 36$$

negative here makes it a hyperbola. Get box!

$$\frac{(y-1)^2}{3^2} - \frac{(x-3)^2}{2^2} = 1$$



Center: $(3,1)$

x width is $2 \cdot 2 = 4$

y width is $2 \cdot 3 = 6$

If $x = 3$, then

$$\frac{(y-1)^2}{3^2} = 1$$

$$y - 1 = \pm 3$$

$$y = 4, -2.$$

vertices $(3,4)$, $(3,-2)$

Get $c = \sqrt{3^2 + 2^2} = \sqrt{13}$

foci: $(3, 1 + \sqrt{13})$
$(3, 1 - \sqrt{13})$