- You should be *expanding* this study guide as you see fit with details and worked examples. With this extra layer of detail you will then have excellent study notes for exams, and later reference.
- Some topics will be emphasized more than others.
- Practice is suggested from Dugopolski *Precalculus, functions and graphs*, 4th Edition.
- Since the expectation is you will take Calculus I at some point, I have included some *Mathematica* syntax where appropriate. *Mathematica* syntax is different than WeBWorK syntax.

Prerequisites: Algebra Review Appendix B

Refer to as needed. We will not be covering these sections in class but will be doing some algebraic manipulations at times that require using these concepts. I suggest reading Appendix B at the start of the semester to reacquaint yourself with the allowable algebraic manipulations. At times we will require some topics from Precalculus I Functions, so I include them here for your benefit.

5.1 Angles and Their Measure (week of Jan 18)

- degree measure, radian measure, and converting between the two (180 degrees $= \pi$ radians)
- acute, obtuse, straight, right, and quadrantal angles
- coterminal angles
- arc length $s = \alpha r$, where α is measured in radians
- area of a sector of a circle is $A = \theta r^2/2$.
- linear velocity $v = s/t$
- angular velocity $\omega = \alpha/t$
- relation between linear and angular velocity is $v = r\omega$
- Practice: 5.1.53-60, 5.1.89-92, 5.1.101-102, 5.1.111, 5.1.114, 5.1.119, 5.1.120
- *Mathematica*: π is `Pi` (capitalization matters!)

5.2 The Sine and Cosine Functions (week of Jan 25)

- review 2.1 Functions
  - domain (x values) and range (y values)
  - functional notation $y = f(x)$
- quadrants I-IV in xy-plane
- unit circle $x^2 + y^2 = 1$
  - quadrant 1
    - special angle $\pi/4$ comes from a square
    - special angles $\pi/3, \pi/6$ comes from an equilateral triangle
  - extending angles from quadrant 1 to other quadrants
- sine and cosine functions definition from the unit circle

  $\sin\alpha = y$
  $\cos\alpha = x$

  Note I tend to use SOH-CAH-TOA from Section 5.6 and reference triangles to work things out rather than trying to memorize the unit circle
- reference angle (used to create a reference triangle on the unit circle)
- fundamental identity (Pythagorean identity): $\cos^2\alpha + \sin^2\alpha = 1$
- Practice: 5.2.5-30, 5.2.31-42, 5.2.51-62, 5.2.101
- *Mathematica*: $\cos x$ is `Cos[x]` and $\sin x$ is `Sin[x]` and $\sin^2 x + \cos^2 x = 1$ is `Cos[x]^2 + Sin[x]^2 ==1`

5.3 The Graphs of Sine and Cosine Functions (WEEK OF FEB 1)

- review 2.3 Families of Functions, Transformations, and Symmetry
  - transformations (graphical and algebraic) of $y = f(x)$ where $h > 0$, $k < 0$, $c > 1$:
    horizontal shift to right: $y = f(x - h)$, $h > 0$
    horizontal shift to left: $y = f(x + h)$, $h > 0$
    vertical shift up: $y = f(x) + k$, $k > 0$
    vertical shift down: $y = f(x) - k$, $k > 0$
    reflect about x axis: $y = -f(x)$
    reflect about y axis: $y = f(-x)$
    stretch vertically: $y = cf(x)$, $c > 1$
    compress horizontally: $y = f(cx)$, $c > 1$
  - notice: anything outside of f is vertical, anything inside of f is horizontal and opposite
  - multiple translations, for example $y = a(x - h)^2 + k$
- a function has period P if $f(x) = f(x + P)$ for all x in domain
- graphs of functions $y = \sin(x)$ and $y = \cos(x)$
  - domain $x \in \mathbb{R}$ and range $y \in [-1, 1]$
  - period $P = 2\pi$
  - amplitude 1
  - fundamental cycle $[0, 2\pi]$
  - sine is odd, $\sin(-x) = -\sin(x)$, and cosine is even, $\cos(-x) = \cos(x)$
- graphs of functions $y = A\sin\big(B(x - C)\big) + D$ and $y = A\cos\big(B(x - C)\big) + D$, assume $B > 0$
  - period $P = 2\pi/B$ (horizontal stretch or compression)
  - amplitude $|A|$ (vertical stretch or compression)
  - phase shift C (horizontal translation)
  - vertical translation D
  - frequency $F = 1/P$ where P is the period of the sine wave
- Practice: 5.3.17-30, 5.3.57, 5.3.61-68, 5.3.71-78
- *Mathematica*: to plot $y = 4\cos(3x + 2)$ where $x \in [-4\pi, 6\pi]$ use `Plot[4*Cos[3x+2],{x,-4Pi,6Pi}]`

5.4 Other Trigonometric Functions and Their Graphs (WEEK OF FEB 8)

- tangent defined from unit circle as $\tan\alpha = \dfrac{y}{x} = \dfrac{\sin\alpha}{\cos\alpha}$
- tangent function $y = \tan(x) = \sin x/\cos x$
  - domain $x \neq \frac{\pi}{2} + k\pi$, k is an integer (vertical asymptotes at these values)
  - range $y \in \mathbb{R}$
  - period is $\pi/2$
  - fundamental cycle is $(-\pi/2, \pi/2)$
  - odd, $\tan(-x) = -\tan(x)$
- cotangent function $y = \cot(x) = \dfrac{\cos x}{\sin x} = \dfrac{1}{\tan x}$
  - domain $x \neq k\pi$, k is an integer (vertical asymptotes at these values)
  - range $y \in \mathbb{R}$
  - period is π
  - fundamental cycle is $(0, \pi)$
  - odd, $\cot(-x) = -\cot(x)$

- secant function $y = \sec(x) = \dfrac{1}{\cos x}$
  - domain $x \neq \frac{\pi}{2} + k\pi$, k is an integer (vertical asymptotes at these values)
  - range $y \in (-\infty, -1] \cup [1, \infty)$
  - period is 2π
  - fundamental cycle is $(0, 2\pi)$
  - even, $\sec(-x) = \sec(x)$
- cosecant function $y = \csc(x) = \dfrac{1}{\sin x}$
  - domain $x \neq k\pi$, k is an integer (vertical asymptotes at these values)
  - range $y \in (-\infty, -1] \cup [1, \infty)$
  - period is 2π
  - fundamental cycle is $(0, 2\pi)$
  - odd, $\csc(-x) = -\csc(x)$
- Practice: 5.4.5-30, 5.4.53-68, 5.4.91
- *Mathematica*: `Tan[x], Cot[x], Sec[x], Csc[x]`

5.5 The Inverse Trig Functions (WEEK OF FEB 15)

- review 2.5 Inverse Functions
  - definition of one-to-one functions (horizontal line test)
  - inverse functions (both graphically, and algebraically)
  - notation for inverse function: $f^{-1}(x) \neq \frac{1}{f(x)}$
  - check using $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$
- inverse sine function $y = \arcsin(x) = \sin^{-1}(x) \neq \frac{1}{\sin x}$
  - domain $[-1, 1]$
  - range $[-\pi/2, \pi/2]$
  - know how to graph
- inverse cosine function $y = \arccos(x) = \cos^{-1}(x) \neq \frac{1}{\cos x}$
  - domain $[-1, 1]$
  - range $[0, \pi]$
  - know how to graph
- inverse cotangent function $y = \arctan(x) = \tan^{-1}(x) \neq \frac{1}{\tan x}$
  - domain $\mathbb{R}$
  - range $(-\pi/2, \pi/2)$
  - know how to graph
- the inverse functions $\sec^{-1}(x)$, $\csc^{-1}(x)$, and $\cot^{-1}(x)$ can also be defined, but are not as commonly used
- compositions of functions should always be simplified

  $\sin(\arcsin(x)) = x$ if $x \in [-1, 1]$
  $\arcsin(\sin(x)) = x$ if $x \in [-\pi/2, \pi/2]$
  $\arcsin(\sin(3\pi/2)) = -\pi/2$ requires some thought to understand
  $\sin(\arctan(x)) = x/\sqrt{x^2 + 1}$ is important in calculus
- Practice: 5.5.37-52, 5.5.69-84, 5.5.101
- *Mathematica*: `ArcSin[x], ArcCos[x], ArcTan[x]`

5.6 Right Triangle Trig (week of Feb 22)

- reference triangles
- SOH-CAH-TOA $\sin\alpha = \dfrac{\text{Opp}}{\text{Hyp}}$, $\cos\alpha = \dfrac{\text{Adj}}{\text{Hyp}}$, $\tan\alpha = \dfrac{\text{Opp}}{\text{Adj}}$
- applications
- Practice: 5.6.13-18, 5.6.27-36, 5.6.45, 5.6.66

Advice: Do not try to memorize all the trig identities. Instead, try to be able to convert from one to another effortlessly. Ultimately, you will only memorize a small number of these identities and will be able to derive others you need from those. I suggest memorizing at least these three:

- $\cos^2 x + \sin^2 x = 1$.
- $\cos(x+y) = \cos x \cos y - \sin x \sin y$.
- $\sin(x+y) = \sin x \cos y + \cos x \sin y$.

6.1 Basic Identities (week of Mar 7)

- an identity is true for all x
- identities are used to create equivalent expressions which involve trig functions
- definition identities

$$\tan x = \frac{\sin x}{\cos x}, \qquad \cot x = \frac{\cos x}{\sin x}, \qquad \sec x = \frac{1}{\cos x}, \qquad \csc x = \frac{1}{\sin x}.$$

- reciprocal identities

$$\sin x = \frac{1}{\csc x}, \qquad \cos x = \frac{1}{\sec x}, \qquad \tan x = \frac{1}{\cot x},$$

$$\csc x = \frac{1}{\sin x}, \qquad \sec x = \frac{1}{\cos x}, \qquad \cot x = \frac{1}{\tan x}.$$

- Pythagorean identities

$$\cos^2 x + \sin^2 x = 1, \qquad 1 + \tan^2 x = \sec^2 x, \qquad \cot^2 x + 1 = \csc^2 x.$$

- even identities

$$\cos(-x) = \cos x, \qquad \sec(-x) = \sec x.$$

- odd identities

$$\sin(-x) = -\sin x, \qquad \tan(-x) = -\tan x, \qquad \csc(-x) = -\csc x, \qquad \cot(-x) = -\cot x.$$

- Practice: 6.1.7-16, 6.1.29-34, 6.1.35-42, 6.1.93-104

6.2 Verifying Identities (week of Mar 7)

- an identity is true for all x
- techniques to verify more complicated trig identities (you might use some or all of these for a given problem):
  - Start with expression on left side, and provide intermediate steps that lead to right side. You could also start with right side and end with left, so generally start with the more complicated expression.
  - Use factoring or multiplying out (ie., algebra) as part of the simplification process.
  - Get common denominators (ie., algebra) as part of the simplification process.
  - Write the expression in terms of sines and cosines.
- Practice: 6.2.55-90

6.3 Sum and Difference Identities (week of Mar 21)

- be able to prove these identities
- use these identities to exactly evaluate trig functions at angles such as

$$-\frac{\pi}{12} = \frac{\pi}{4} - \frac{\pi}{3}, \qquad \frac{7\pi}{12} = \frac{\pi}{4} + \frac{\pi}{3}, \qquad \frac{\pi}{12} = \frac{\pi}{4} - \frac{\pi}{6}, \qquad \frac{5\pi}{12} = \frac{\pi}{4} + \frac{\pi}{6}, \qquad \text{etc.}$$

- sum/difference identities

$$\begin{aligned} \cos(u+v) &= \cos u \cos v - \sin u \sin v, \qquad & \sin(u+v) &= \sin u \cos v + \cos u \sin v, \\ \cos(u-v) &= \cos u \cos v + \sin u \sin v, \qquad & \sin(u-v) &= \sin u \cos v - \cos u \sin v. \end{aligned}$$

- sum/difference identities

$$\tan(u+v) = \frac{\tan u + \tan v}{1 - \tan u \tan v},$$

$$\tan(u-v) = \frac{\tan u - \tan v}{1 + \tan u \tan v}.$$

- cofunction identities

$$\sin\left(\frac{\pi}{2} - u\right) = \cos u, \qquad \cos\left(\frac{\pi}{2} - u\right) = \sin u, \qquad \tan\left(\frac{\pi}{2} - u\right) = \cot u,$$

$$\sec\left(\frac{\pi}{2} - u\right) = \csc u, \qquad \csc\left(\frac{\pi}{2} - u\right) = \sec u, \qquad \cot\left(\frac{\pi}{2} - u\right) = \tan u.$$

- Practice: 6.3.13-20, 6.3.37-48, 6.3.65, 6.3.83-100

6.4 Double-Angle and Half-Angle Identities (week of Mar 28)

- derived from sum and difference identities
- double-angle identities

$$\sin(2x) = 2\sin x\cos x, \qquad \tan(2x) = \frac{2\tan x}{1-\tan^2 x},$$

$$\cos(2x) = \cos^2 x - \sin^2 x, \qquad \cos(2x) = 2\cos^2 x - 1, \qquad \cos(2x) = 1 - 2\sin^2 x.$$

- half-angle identities

$$\cos^2 x = \frac{1+\cos(2x)}{2} \quad \rightarrow \quad \cos\left(\frac{x}{2}\right) = \pm\sqrt{\frac{1+\cos x}{2}},$$

$$\sin^2 x = \frac{1-\cos(2x)}{2} \quad \rightarrow \quad \sin\left(\frac{x}{2}\right) = \pm\sqrt{\frac{1-\cos x}{2}},$$

- half-angle identities for tangent

$$\tan\left(\frac{x}{2}\right) = \pm\sqrt{\frac{1-\cos x}{1+\cos x}}, \qquad \tan\left(\frac{x}{2}\right) = \frac{\sin x}{1+\cos x}, \qquad \tan\left(\frac{x}{2}\right) = \frac{1-\cos x}{\sin x}.$$

- Practice: 6.4.23-34, 6.4.67-74, 6.4.75

6.5 Product and Sum Identities (week of Apr 4)

- these identities are not as important as the previous identities (do not memorize them)
- arrived at by adding sine and cosine sum/difference identities
- product-to-sum identities

$$\begin{aligned}
\sin u\cos v &= \frac{1}{2}\big[\sin(u+v) + \sin(u-v)\big],\\
\sin u\sin v &= \frac{1}{2}\big[\cos(u-v) - \cos(u+v)\big],\\
\cos u\cos v &= \frac{1}{2}\big[\cos(u-v) + \cos(u+v)\big].
\end{aligned}$$

- sum-to-product identities (do not memorize, look them up if you need them)
- reduction formula $a\sin x + b\cos x = \sqrt{a^2+b^2}\sin(x+\alpha)$ where α is an angle whose terminal side contains the point (a, b)
- Practice: 6.5.29-38

6.6 Conditional Trig Equations (week of Apr 4)

- this section is about using trig identities and algebra to solve equations
- conditional equations have at least one solution
- trig equations will have an infinite number of solutions due to periodicity of trig functions
- find all real numbers that satisfy $\cos x = \sqrt{3}/2$ (solution $x = \pm\pi/6 + 2n\pi$)
- find all real numbers that satisfy $\sin x = \sqrt{3}/2$ (solution $x = 2\pi/3 + 2n\pi$ and $x = \pi/3 + 2n\pi$)
- find all real numbers that satisfy $\tan x = \sqrt{3}/2$
- more complicated equations may involve more algebra. Review 3.4 Miscellaneous Equations
  - factoring higher degree equations
  - equations involving square roots
  - equations with rational exponents
  - equations of quadratic type
  - equations involving absolute values
- Practice: 6.6.1-18, 6.6.65, 6.6.66, 6.6.68, 6.6.84

7.1 Law of Sines (week of Apr 18)

- an oblique triangle is an triangle without a right angle
- oblique triangle with sides a, b, c and angles α, β, γ, then the Law of Sines is

$$\frac{\sin\alpha}{a} = \frac{\sin\beta}{b} = \frac{\sin\gamma}{c}$$

- cases: ASA (angle-side-angle) and SSA (side-side-angle, which is ambiguous)
- area of triangle

$$A = \frac{1}{2}bc\sin\alpha = \frac{1}{2}ac\sin\beta = \frac{1}{2}ab\sin\gamma$$

- area of quadrilateral
- Practice: 7.1.3-6, 7.1.38

7.2 Law of Cosines (week of Apr 18)

- oblique triangle with sides a, b, c and angles α, β, γ, then the Law of Cosines is

$$a^2 = b^2 + c^2 - 2bc\cos\alpha,$$
$$b^2 = a^2 + c^2 - 2ac\cos\beta,$$
$$c^2 = a^2 + b^2 - 2ab\cos\gamma.$$

- understand the proof
- cases: SSS (side-side-side) and SAS (side-angle-side)
- area of triangle by Heron's formula

$$A = \sqrt{A(S-a)(S-b)(S-c)}, \quad \text{where } S = (a+b+c)/2$$

- length of chord
- Practice: 7.2.7-16, 7.2.18-20, 7.2.33-38

Depending on time, these topics will be covered briefly. All these topics are interesting and useful, but will not be necessary for calculus. They may show up in later math classes (for example, we use Euler's formula extensively in differential equations, and understanding polar coordinates would be useful in Calculus III).

7.4 Trig Form of Complex Numbers (week of Apr 25)

- the complex plane (real axis, imaginary axis)
- $\left|a+bi\right| = \sqrt{a^2+b^2}$
- $z = a+bi = r(\cos\theta + i\sin\theta)$ where $r=\sqrt{a^2+b^2}$ and θ is an angle in standard position whose terminal side contains the point (a,b)
- complex conjugate of $a+bi$ is $a-bi$
- algebra (why this representation is useful)

$$z_1 z_2 = r_1 r_2 \left[\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\right]$$
$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\left[\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)\right]$$

- Practice: 7.4.55-58, 7.4.75
- *Mathematica*: `a+b*I`

7.5 Powers and Roots of Complex Numbers (week of Apr 25)

- De Moivre's Theorem $z^n = r^n(\cos(n\theta)+i\sin(n\theta))$ where n is a positive integer
- Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$
- nth root of a complex number using Euler's formula
- Practice: 7.5.11-16, 7.5.29
- *Mathematica*: `Exp[I*t], Cos[t]+I*Sin[t]`

7.6 Polar Equations (week of May 2)

- polar coordinates $x = r\cos\theta, \quad y = r\sin\theta$

  $r = \sqrt{x^2+y^2}$

  θ is any angle in standard position whose terminal side contains (x,y)

- polar equation $r=f(\theta)$ (compare to cartesian equation $y=f(x)$)
- converting from polar equation to cartesian equation
- Practice: 7.6.77-88
- *Mathematica*: `PolarPlot[t,{t,0,2Pi}]`