This is not a complete list of the types of problems to expect on the final exam.
The final exam will have 8 questions worth 20 marks each.
The may be some True/False on the exam.

Example 1 Find $\csc\theta$ if $\tan\theta = \frac{77}{2}$ and $\sin\theta < 0$.

Example 2 Find an algebraic expression equivalent to the expression $\sin\left(\arccos\left(\frac{1}{x}\right)\right)$.

Example 3 Solve $\cos 2x + \cos x = 0$ algebraically for exact solutions in the interval $[0, 2\pi)$.

Example 4 Find the value of $\sin\left(\frac{\pi}{12}\right)$ exactly using an angle difference formula.

Example 5 Use the power reducing identities to prove the identity $\sin^4 x = \frac{1}{8}(3 - 4\cos 2x + \cos 4x)$.

Example 6 Convert the rectangular equation $(x+3)^2 + (y+3)^2 = 18$ to a polar equation.

Example 7 Starting from $\cos(u - v) = \cos u \cos v + \sin u \sin v$, derive an expression for $\sin(u + v)$.

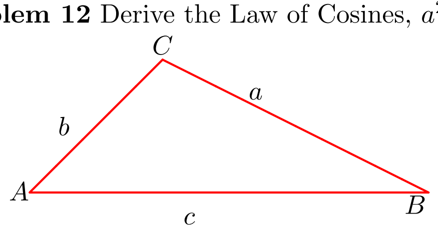
Example 8 Starting from $\cos(u + v) = \cos u \cos v - \sin u \sin v$, prove the identity $\cos^2 u = \frac{1 + \cos 2u}{2}$.

Example 9 Prove the identity $\sec 2u = \frac{\sec^2 u}{2 - \sec^2 u}$.

Example 10 Solve $\tan(x/2) = \sin x$ for $x \in [0, \pi)$.

Example 11 Show why $\tan\left(-\frac{\pi}{12}\right) = -2 + \sqrt{3}$ using angle difference formulas.

Problem 12 Derive the Law of Cosines, $a^2 = b^2 + c^2 - 2bc\cos A$, given the triangle



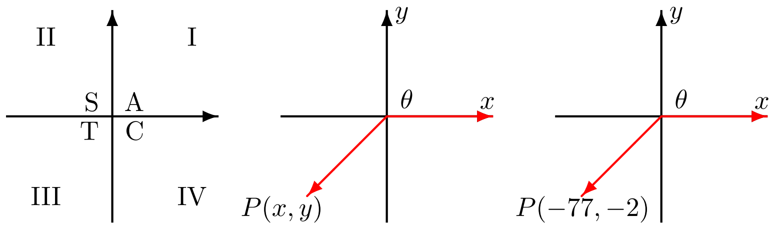
Example 13 Draw a well-labeled sketch that shows why the solutions to $\sin x = M$ (where $0 < M < 1$) are $x = \arcsin M \pm 2n\pi$ and $x = \pi - \arcsin M \pm 2n\pi$, $n = 0, 1, 2, 3, \ldots$.

Example 14 Draw a well-labeled sketch of $y = \tan x$ (include two periods of the function $y = \tan x$) and $y = \arctan x$.

Solutions

Example 1 Find $\csc\theta$ if $\tan\theta = \dfrac{77}{2}$ and $\sin\theta < 0$.

Since $\sin\theta$ is less than zero, we must be in either Quadrant III or IV.
Since $\tan\theta$ is greater than zero we must be in either Quadrant I or III.
Therefore, the angle θ has a terminal side in Quadrant III.



Since $\tan\theta = \dfrac{y}{x} = \dfrac{77}{2} = \dfrac{-77}{-2}$, we have $y = -77$, $x = -2$.

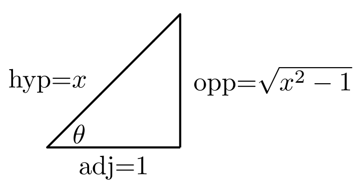
The distance $r = \sqrt{x^2+y^2} = \sqrt{(-77)^2+(-2)^2} = \sqrt{5933}$.

$$\text{Therefore, } \csc\theta = \frac{1}{\sin\theta} = \frac{r}{y} = \frac{\sqrt{5933}}{-77} = -\frac{\sqrt{5933}}{77}.$$

Example 2 Find an algebraic expression equivalent to the expression $\sin\left(\arccos\left(\frac{1}{x}\right)\right)$.

To simplify this let $\theta = \arccos\left(\frac{1}{x}\right)$. This means $\cos\theta = \dfrac{1}{x} = \dfrac{\text{adj}}{\text{hyp}}$.

Construct a reference triangle



The length of the opposite side was found using the Pythagorean theorem

$$\sin\left(\arccos\left(\frac{1}{x}\right)\right) = \sin\theta = \frac{\text{opp}}{\text{hyp}} = \frac{\sqrt{x^2-1}}{x}.$$

Example 3 Solve $\cos 2x + \cos x = 0$ algebraically for exact solutions in the interval $[0, 2\pi)$.

$$\begin{aligned}
\cos 2x + \cos x &= \cos^2 x - \sin^2 x + \cos x \\
&= \cos^2 x - (1 - \cos^2 x) + \cos x \\
&= 2\cos^2 x + \cos x - 1 = 0
\end{aligned}$$

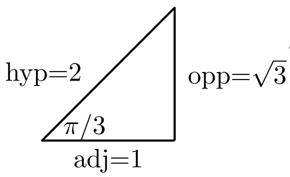
Let $y = \cos x$. Then

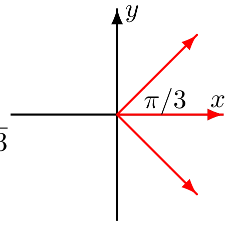
$$\begin{aligned}
\cos 2x + \cos x &= 2\cos^2 x + \cos x - 1 = 0 \\
&= 2y^2 + y - 1 = 0 \\
y &= \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{-1 \pm \sqrt{1+8}}{4} = \frac{-1 \pm 3}{4} = \frac{2}{4} \text{ or } \frac{-4}{4} = \frac{1}{2} \text{ or } -1
\end{aligned}$$

So we must solve $y = \cos x = 1/2$ and $y = \cos x = -1$.

The equation $\cos x = -1$ has a solution of π in the interval $[0, 2\pi)$.

The equation $\cos x = \text{adj/hyp} = 1/2$ corresponds to one of our special triangles:





So the solution is $\pi/3$.

There is also a solution in Quadrant IV at $2\pi - \pi/3 = 5\pi/3$ in the interval $[0, 2\pi)$.

The solutions to $\cos 2x + \cos x = 0$ in the interval $[0, 2\pi)$ are $\dfrac{\pi}{3}, \pi, \dfrac{5\pi}{3}$.

Example 4 Find the value of $\sin\left(\dfrac{\pi}{12}\right)$ exactly using an angle difference formula.

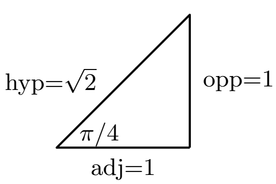
Solution:

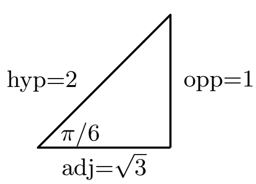
First, we need to figure out how to relate $\pi/12$ to some of our special angles, since we are told to find this answer exactly.

$$\frac{\pi}{12} = \frac{2\pi}{24} = \frac{6\pi - 4\pi}{24} = \frac{\pi}{4} - \frac{\pi}{6}.$$

Therefore,

$$\begin{aligned}
\sin\left(\frac{\pi}{12}\right) &= \sin\left(\frac{\pi}{4} - \frac{\pi}{6}\right) \\
&= \sin\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{6}\right) - \cos\left(\frac{\pi}{4}\right)\sin\left(\frac{\pi}{6}\right), \qquad \text{use } \sin(u - v) = \sin u \cos v - \cos u \sin v \\
&= \left(\frac{1}{\sqrt{2}}\right)\left(\frac{\sqrt{3}}{2}\right) - \left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{2}\right), \qquad \text{using reference triangles below} \\
&= \frac{\sqrt{3}}{2\sqrt{2}} - \frac{1}{2\sqrt{2}} = \frac{\sqrt{3} - 1}{2\sqrt{2}}
\end{aligned}$$





Example 5 Use the power reducing identities to prove the identity $\sin^4 x = \dfrac{1}{8}(3 - 4\cos 2x + \cos 4x)$.

Solution:

$$\begin{aligned}
\sin^4 x &= \left(\sin^2 x\right)^2 \\
&= \left(\frac{1-\cos 2x}{2}\right)^2, \quad \text{using } \sin^2 u = \frac{1-\cos 2u}{2}, \text{ with } u = x. \\
&= \frac{1}{4}\left(1-\cos 2x\right)^2 \\
&= \frac{1}{4}\left(1+\cos^2 2x - 2\cos 2x\right) \\
&= \frac{1}{4}\left(1+\left(\frac{1+\cos 4x}{2}\right) - 2\cos 2x\right), \quad \text{using } \cos^2 u = \frac{1+\cos 2u}{2}, \text{ with } u = 2x. \\
&= \frac{1}{4}\left(\frac{2}{2}+\frac{1+\cos 4x}{2} - \frac{4\cos 2x}{2}\right) \\
&= \frac{1}{8}\left(2+1+\cos 4x - 4\cos 2x\right) \\
&= \frac{1}{8}\left(3+\cos 4x - 4\cos 2x\right) \\
&= \frac{1}{8}\left(3-4\cos 2x + \cos 4x\right)
\end{aligned}$$

Example 6 Convert the rectangular equation $(x+3)^2 + (y+3)^2 = 18$ to a polar equation.

We simply use our relations:

$$\begin{aligned}
x &= r\cos\theta \\
y &= r\sin\theta,
\end{aligned}$$

$$\begin{aligned}
(x+3)^2 + (y+3)^2 &= 18 \\
(r\cos\theta+3)^2 + (r\sin\theta+3)^2 &= 18 \\
(r^2\cos^2\theta + 9 + 6r\cos\theta) + (r^2\sin^2\theta + 9 + 6r\sin\theta) &= 18 \\
r^2(\cos^2\theta + \sin^2\theta) + 18 + 6r\cos\theta + 6r\sin\theta &= 18 \\
r^2(1) + 6r\cos\theta + 6r\sin\theta &= 18 - 18 \\
r^2 + 6r\cos\theta + 6r\sin\theta &= 0 \\
r(r + 6\cos\theta + 6\sin\theta) &= 0 \\
r + 6\cos\theta + 6\sin\theta &= 0, \qquad r \neq 0 \\
r &= -6\cos\theta - 6\sin\theta
\end{aligned}$$

Example 7 Starting from $\cos(u-v) = \cos u\cos v + \sin u \sin v$, derive an expression for $\sin(u+v)$.

$$\begin{aligned}
\cos(u-v) &= \cos u\cos v + \sin u\sin v \\
\sin(u+v) &= \cos\left(\frac{\pi}{2} - (u+v)\right) \\
&= \cos\left(\frac{\pi}{2} - u - v\right) \\
&= \cos\left(\left(\frac{\pi}{2} - u\right) - v\right) \\
&= \cos\left(\frac{\pi}{2} - u\right)\cos v + \sin\left(\frac{\pi}{2} - u\right)\sin v \\
&= \sin u\cos v + \cos u\sin v
\end{aligned}$$

Example 8 Starting from $\cos(u+v) = \cos u \cos v - \sin u \sin v$, prove the identity $\cos^2 u = \dfrac{1+\cos 2u}{2}$.

$$\begin{aligned}
\cos(u+v) &= \cos u \cos v - \sin u \sin v \\
\cos(2u) = \cos(u+u) &= \cos u \cos u - \sin u \sin u \\
&= \cos^2 u - \sin^2 u \\
&= \cos^2 u - (1 - \cos^2 u) \\
\cos 2u &= 2\cos^2 u - 1 \\
\cos^2 u &= \frac{1+\cos 2u}{2}
\end{aligned}$$

Example 9 Prove the identity $\sec 2u = \dfrac{\sec^2 u}{2 - \sec^2 u}$.

$$\sec 2u = \frac{1}{\cos 2u}$$

Pause to figure out the trig identity we need.

$$\begin{aligned}
\cos(u-v) &= \cos u \cos v + \sin u \sin v \\
\cos(u+v) = \cos(u-(-v)) &= \cos u \cos(-v) + \sin u \sin(-v) \\
&= \cos u \cos v - \sin u \sin v \\
\cos(2u) &= \cos^2 u - \sin^2 u
\end{aligned}$$

Back to our problem:

$$\begin{aligned}
\sec 2u &= \frac{1}{\cos 2u} \\
&= \frac{1}{\cos^2 u - \sin^2 u} \\
&= \frac{1}{\cos^2 u - \sin^2 u} \cdot \left(\frac{\sec^2 u}{\sec^2 u}\right) \\
&= \frac{\sec^2 u}{(\cos^2 u - \sin^2 u)\sec^2 u} \\
&= \frac{\sec^2 u}{(\cos^2 u - \sin^2 u)\frac{1}{\cos^2 u}} \\
&= \frac{\sec^2 u}{1 - \tan^2 u}
\end{aligned}$$

Pause to figure out the trig identity we need.

$$\begin{aligned}
\cos^2 x + \sin^2 x &= 1 \\
\frac{\cos^2 x}{\cos^2 x} + \frac{\sin^2 x}{\cos^2 x} &= \frac{1}{\cos^2 x} \\
1 + \tan^2 x &= \sec^2 x \\
\tan^2 x &= \sec^2 x - 1
\end{aligned}$$

Back to our problem:

$$\begin{aligned}
\sec 2u &= \frac{\sec^2 u}{1 - \tan^2 u} \\
&= \frac{\sec^2 u}{1 - (\sec^2 u - 1)} \\
\sec 2u &= \frac{\sec^2 u}{2 - \sec^2 u}
\end{aligned}$$

Example 10 Solve $\tan(x/2) = \sin x$ for $x \in [0, \pi)$.

We need to convert the half angle tangent function to trig functions of x.

Pause to work out some trig identities:

$$\begin{aligned}
\cos(u-v) &= \cos u \cos v + \sin u \sin v \\
\cos(u+v) = \cos(u-(-v)) &= \cos u \cos(-v) + \sin u \sin(-v) \\
&= \cos u \cos v - \sin u \sin v \\
\cos(2u) &= \cos^2 u - \sin^2 u \\
&= \cos^2 u - (1 - \cos^2 u) \\
&= 2\cos^2 u - 1 \\
\cos^2 u &= \frac{1}{2}(1 + \cos 2u)
\end{aligned}$$

$$\begin{aligned}
\cos(2u) &= \cos^2 u - \sin^2 u \\
&= (1 - \sin^2 u) - 1 \\
&= 1 - 2\sin^2 u \\
\sin^2 u &= \frac{1}{2}(1 - \cos 2u)
\end{aligned}$$

$$\begin{aligned}
\tan^2 u &= \frac{\sin^2 u}{\cos^2 u} \\
&= \frac{1 - \cos 2u}{1 + \cos 2u} \\
&= \frac{1 - \cos 2u}{1 + \cos 2u} \cdot \left(\frac{1 - \cos 2u}{1 - \cos 2u}\right) \\
&= \frac{(1 - \cos 2u)^2}{1 - \cos^2 2u} \\
&= \frac{(1 - \cos 2u)^2}{\sin^2 2u} \\
\tan^2 u &= \left(\frac{1 - \cos 2u}{\sin 2u}\right)^2 \\
\tan u &= \frac{1 - \cos 2u}{\sin 2u}
\end{aligned}$$

The last line is true since $\sin 2u$ and $\tan u$ have the same sign at any point.

This was a serious amount of work, but look at how many trig identities we found along the way! On a test, these identities can be reused in other problems if needed. This is probably the most work you would ever have to do so derive certain trig identities; most of the time the derivation will be significantly shorter.

Now we can work on our problem:

$$\begin{aligned}
\tan(x/2) &= \sin x \\
\frac{1 - \cos x}{\sin x} &= \sin x, \quad \text{(above formula with } u = x/2\text{)} \\
1 - \cos x &= \sin^2 x \\
1 - \cos x &= 1 - \cos^2 x \\
-\cos x &= -\cos^2 x \\
\cos^2 x - \cos x &= 0 \\
\cos x(\cos x - 1) &= 0
\end{aligned}$$

So we need to solve $\cos x = 0$ and $\cos x - 1 = 0$.

For the first, $\cos x = 0$ for $x = \pi/2 \in [0, \pi)$.
For the second, $\cos x = 1$ for $x = 0 \in [0, \pi)$.

The two solutions are $x = 0, \dfrac{\pi}{2}$ for $x \in [0, \pi)$.

Example 11 Show why $\tan\left(-\dfrac{\pi}{12}\right) = -2 + \sqrt{3}$ using angle difference formulas.

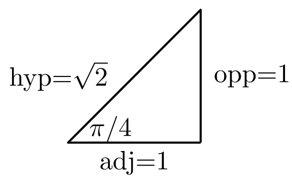
We can write the tangent in terms of sine and cosine functions:

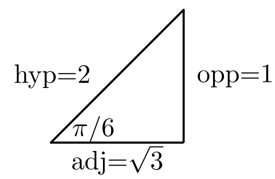
$$\tan\left(-\frac{\pi}{12}\right) = \frac{\sin\left(-\frac{\pi}{12}\right)}{\cos\left(-\frac{\pi}{12}\right)}.$$

Now, we need to figure out how to relate $-\pi/12$ to some of our special angles, since we are told to find this answer exactly.

$$\frac{-\pi}{12} = \frac{-2\pi}{24} = \frac{4\pi - 6\pi}{24} = \frac{\pi}{6} - \frac{\pi}{4}.$$

Here are the reference triangles we will need:





We need cosine and sine of a difference identities, which are

$$\begin{aligned}
\cos(u - v) &= \cos u \cos v + \sin u \sin v \text{ (memorized)} \\
\sin(u + v) &= \sin u \cos v + \cos u \sin v \text{ (memorized)} \\
\sin(u - v) &= \sin(u + (-v)) \text{ (work this out, using above identity)} \\
&= \sin u \cos(-v) + \cos u \sin(-v) \\
\sin(u - v) &= \sin u \cos v - \cos u \sin v \text{ (since cosine is even and sine is odd)}
\end{aligned}$$

We have what we need to solve the problem.

Therefore,

$$\begin{aligned}
\sin\left(-\frac{\pi}{12}\right) &= \sin\left(\frac{\pi}{6} - \frac{\pi}{4}\right) \\
&= \sin\left(\frac{\pi}{6}\right)\cos\left(\frac{\pi}{4}\right) - \cos\left(\frac{\pi}{6}\right)\sin\left(\frac{\pi}{4}\right), \quad \text{use } \sin(u - v) = \sin u \cos v - \cos u \sin v \\
&= \left(\frac{1}{2}\right)\left(\frac{1}{\sqrt{2}}\right) - \left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{\sqrt{2}}\right), \quad \text{using reference triangles above} \\
&= \frac{1}{2\sqrt{2}} - \frac{\sqrt{3}}{2\sqrt{2}} = \frac{1 - \sqrt{3}}{2\sqrt{2}}
\end{aligned}$$

and

$$\begin{aligned}
\cos\left(-\frac{\pi}{12}\right) &= \cos\left(\frac{\pi}{6} - \frac{\pi}{4}\right) \\
&= \cos\left(\frac{\pi}{6}\right)\cos\left(\frac{\pi}{4}\right) + \sin\left(\frac{\pi}{6}\right)\sin\left(\frac{\pi}{4}\right), \quad \text{use } \cos(u - v) = \cos u \cos v + \sin u \sin v \\
&= \left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{\sqrt{2}}\right) + \left(\frac{1}{2}\right)\left(\frac{1}{\sqrt{2}}\right), \quad \text{using reference triangles above} \\
&= \frac{\sqrt{3}}{2\sqrt{2}} + \frac{1}{2\sqrt{2}} = \frac{\sqrt{3} + 1}{2\sqrt{2}}
\end{aligned}$$

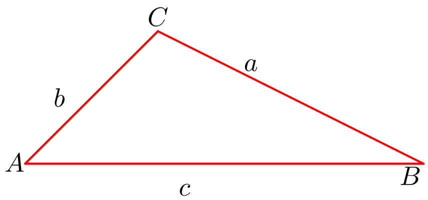
So we have

$$\tan\left(-\frac{\pi}{12}\right) = \frac{\sin\left(-\frac{\pi}{12}\right)}{\cos\left(-\frac{\pi}{12}\right)} = \left(\frac{1-\sqrt{3}}{2\sqrt{2}}\right) \times \left(\frac{2\sqrt{2}}{\sqrt{3}+1}\right) = \frac{1-\sqrt{3}}{1+\sqrt{3}}.$$

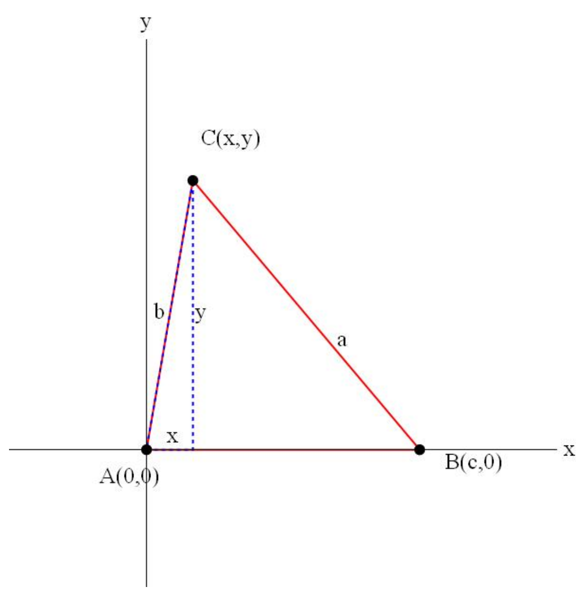
To get the final result asked for, we can rationalize the denominator:

$$\tan\left(-\frac{\pi}{12}\right) = \frac{1-\sqrt{3}}{1+\sqrt{3}} = \frac{1-\sqrt{3}}{1+\sqrt{3}} \times \left(\frac{1-\sqrt{3}}{1-\sqrt{3}}\right) = \frac{1-2\sqrt{3}+3}{1-3} = \frac{4-2\sqrt{3}}{-2} = -2+\sqrt{3}.$$

Problem 12 Derive the Law of Cosines, $a^2 = b^2 + c^2 - 2bc\cos A$, given the triangle



The law of cosines is a generalization of the Pythagorean theorem. It can be derived in a manner similar to how we derived the formula for $\cos(u - v)$. Let's introduce a coordinate system (my triangle has changed in scale, but otherwise the edges a, b, and c all line up!):



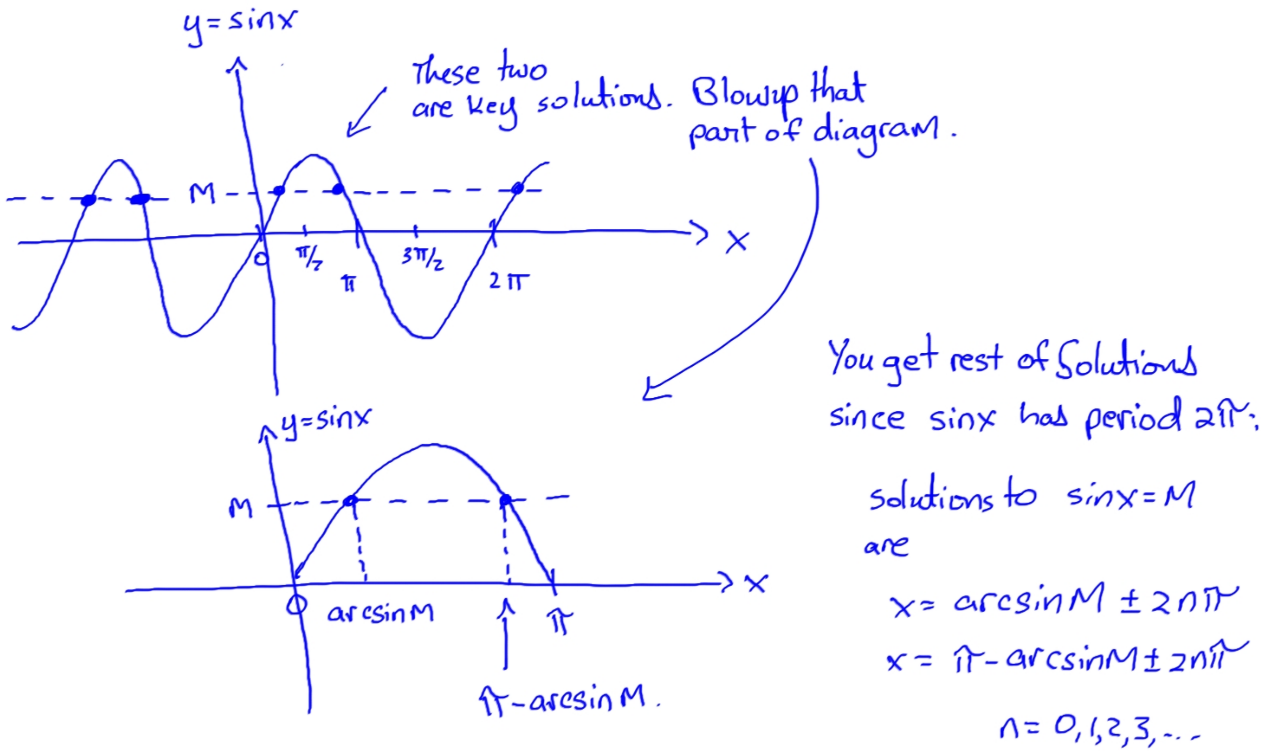
The coordinates of the point C satisfy:

$$\frac{x}{b} = \cos A \quad \text{and} \quad \frac{y}{b} = \sin A$$

Therefore, $x = b\cos A$ and $y = b\sin A$. Using the distance formula, we can write for the distance from point C to B:

$$\begin{aligned}
a &= \sqrt{(x-c)^2 + (y-0)^2} \\
a^2 &= (x-c)^2 + y^2 \\
a^2 &= (b\cos A - c)^2 + (b\sin A)^2 \\
a^2 &= b^2\cos^2 A + c^2 - 2bc\cos A + b^2\sin^2 A \\
a^2 &= b^2(\cos^2 A + \sin^2 A) + c^2 - 2bc\cos A \\
a^2 &= b^2(1) + c^2 - 2bc\cos A \\
a^2 &= b^2 + c^2 - 2bc\cos A
\end{aligned}$$

Example 13 Draw a well-labeled sketch that shows why the solutions to $\sin x = M$ (where $0 < M < 1$) are $x = \arcsin M \pm 2n\pi$ and $x = \pi - \arcsin M \pm 2n\pi$, $n = 0, 1, 2, 3, \ldots$.



Example 14 Draw a well-labeled sketch of $y = \tan x$ (include two periods of the function $y = \tan x$) and $y = \arctan x$.