The formulas I have memorized:

$\boxed{\cos^2 x + \sin^2 x = 1}$

$\boxed{\cos(u - v) = \cos u \cos v + \sin u \sin v}$

$\boxed{\sin(u + v) = \sin u \cos v + \cos u \sin v}$

Any other formula I need I will derive from these.

Showing the steps of your solution will be critical on this test.

Questions

1. Show why $\tan\left(-\frac{\pi}{12}\right) = -2 + \sqrt{3}$ using angle difference formulas.

2. Solve $\sin 2x + \sin 4x = 0$ exactly for all solutions in the interval $[0, 2\pi)$.

3. (challenging) Prove the identity $\sec 2u = \dfrac{\sec^2 u}{2 - \sec^2 u}$.

4. Prove the trig identity $\cos(u - v) = \cos u \cos v + \sin u \sin v$ using the following process:
Draw a diagram of the unit circle which includes the angles u, v, and $\theta = u - v$.
Draw a second diagram with the angle θ in standard position.
Use the distance formula and simplify to get the given identity.

Solutions

1. Show why $\tan\left(-\frac{\pi}{12}\right) = -2+\sqrt{3}$ using angle difference formulas.

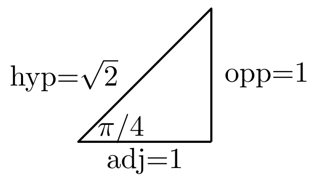
We can write the tangent in terms of sine and cosine functions:

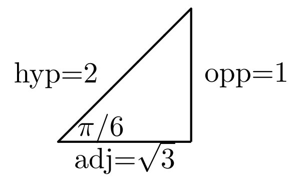
$$\tan\left(-\frac{\pi}{12}\right) = \frac{\sin\left(-\frac{\pi}{12}\right)}{\cos\left(-\frac{\pi}{12}\right)}.$$

Now, we need to figure out how to relate $-\pi/12$ to some of our special angles, since we are told to find this answer exactly.

$$\frac{-\pi}{12} = \frac{-2\pi}{24} = \frac{4\pi-6\pi}{24} = \frac{\pi}{6} - \frac{\pi}{4}.$$

Here are the reference triangles we will need:





We need cosine and sine of a difference identities, which are

$$\begin{aligned}
\cos(u-v) &= \cos u\cos v + \sin u \sin v \text{ (memorized)}\\
\sin(u+v) &= \sin u\cos v + \cos u \sin v \text{ (memorized)}\\
\sin(u-v) &= \sin(u+(-v)) \text{ (work this out, using above identity)}\\
&= \sin u\cos(-v) + \cos u \sin(-v)\\
\sin(u-v) &= \sin u\cos v - \cos u \sin v \text{ (since cosine is even and sine is odd)}
\end{aligned}$$

We have what we need to solve the problem.

Therefore,

$$\begin{aligned}
\sin\left(-\frac{\pi}{12}\right) &= \sin\left(\frac{\pi}{6}-\frac{\pi}{4}\right)\\
&= \sin\left(\frac{\pi}{6}\right)\cos\left(\frac{\pi}{4}\right) - \cos\left(\frac{\pi}{6}\right)\sin\left(\frac{\pi}{4}\right), \quad \text{use } \sin(u-v) = \sin u\cos v - \cos u \sin v\\
&= \left(\frac{1}{2}\right)\left(\frac{1}{\sqrt{2}}\right) - \left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{\sqrt{2}}\right), \quad \text{using reference triangles above}\\
&= \frac{1}{2\sqrt{2}} - \frac{\sqrt{3}}{2\sqrt{2}} = \frac{1-\sqrt{3}}{2\sqrt{2}}
\end{aligned}$$

and

$$\begin{aligned}
\cos\left(-\frac{\pi}{12}\right) &= \cos\left(\frac{\pi}{6}-\frac{\pi}{4}\right)\\
&= \cos\left(\frac{\pi}{6}\right)\cos\left(\frac{\pi}{4}\right) + \sin\left(\frac{\pi}{6}\right)\sin\left(\frac{\pi}{4}\right), \quad \text{use } \cos(u-v) = \cos u\cos v + \sin u \sin v\\
&= \left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{\sqrt{2}}\right) + \left(\frac{1}{2}\right)\left(\frac{1}{\sqrt{2}}\right), \quad \text{using reference triangles above}\\
&= \frac{\sqrt{3}}{2\sqrt{2}} + \frac{1}{2\sqrt{2}} = \frac{\sqrt{3}+1}{2\sqrt{2}}
\end{aligned}$$

So we have

$$\tan\left(-\frac{\pi}{12}\right) = \frac{\sin\left(-\frac{\pi}{12}\right)}{\cos\left(-\frac{\pi}{12}\right)} = \left(\frac{1-\sqrt{3}}{2\sqrt{2}}\right) \times \left(\frac{2\sqrt{2}}{\sqrt{3}+1}\right) = \frac{1-\sqrt{3}}{1+\sqrt{3}}.$$

To get the final result asked for, we can rationalize the denominator:

$$\tan\left(-\frac{\pi}{12}\right) = \frac{1-\sqrt{3}}{1+\sqrt{3}} = \frac{1-\sqrt{3}}{1+\sqrt{3}} \times \left(\frac{1-\sqrt{3}}{1-\sqrt{3}}\right) = \frac{1-2\sqrt{3}+3}{1-3} = \frac{4-2\sqrt{3}}{-2} = -2+\sqrt{3}.$$

2. Solve $\sin 2x + \sin 4x = 0$ exactly for all solutions in the interval $[0, 2\pi)$.

The main issue is the angles are different. Let's use:

$$\sin 4x = \sin(2x+2x) = \sin 2x \cos 2x + \sin 2x \cos 2x = 2\sin 2x \cos 2x$$

to get all the angles being the same and then see what happens.

$$\begin{aligned} \sin 2x + \sin 4x &= \sin 2x + 2\sin 2x \cos 2x, \quad . \\ &= \sin 2x(1+2\cos 2x) = 0 \\ & \quad \sin 2x = 0 \quad \text{or} \quad 1+2\cos 2x = 0 \\ & \quad \sin y = 0 \quad \text{or} \quad 1+2\cos y = 0 \end{aligned}$$

Where we have let $y = 2x$. Since we want $x \in [0, 2\pi)$, we should search for all solutions $y \in [0, 4\pi)$.

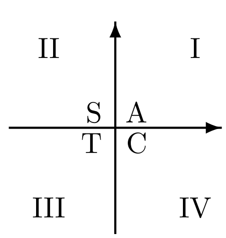
First, $\sin y = 0$ if $y = 0, \pi, 2\pi, 3\pi$. These are all the solutions for $y \in [0, 4\pi)$.

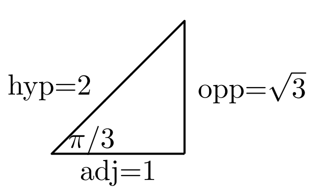
$$\begin{aligned} y = 2x = 0 &\longrightarrow x = 0 \\ y = 2x = \pi &\longrightarrow x = \frac{\pi}{2} \\ y = 2x = 2\pi &\longrightarrow x = \pi \\ y = 2x = 3\pi &\longrightarrow x = \frac{3\pi}{2} \end{aligned}$$

Now, $1 + 2\cos y = 0$, which means $\cos y = -\dfrac{1}{2}$.

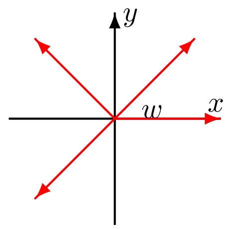
Since the cosine is negative, we must be in either Quadrant II or III.

Let's figure out what the solution to $\cos w = \text{adj}/\text{hyp} = \dfrac{1}{2}$ is. This comes from one of our special triangles:





So $w = \pi/3$. We want the corresponding solutions in Quadrants II and III.



$$\begin{aligned} y &= \pi - w = \frac{2\pi}{3} \\ y &= \pi + w = \frac{4\pi}{3} \end{aligned}$$

Now we need all the solutions $y \in [0, 4\pi)$:

$$\begin{aligned} y &= \frac{2\pi}{3} \\ y &= \frac{4\pi}{3} \\ y &= \frac{2\pi}{3} + 2\pi = \frac{8\pi}{3} \\ y &= \frac{4\pi}{3} + 2\pi = \frac{10\pi}{3} \end{aligned}$$

Now we need the solutions we seek, x:

$$\begin{aligned} y = 2x = \frac{2\pi}{3} &\longrightarrow x = \frac{\pi}{3} \\ y = 2x = \frac{4\pi}{3} &\longrightarrow x = \frac{2\pi}{3} \\ y = 2x = \frac{8\pi}{3} &\longrightarrow x = \frac{4\pi}{3} \\ y = 2x = \frac{10\pi}{3} &\longrightarrow x = \frac{5\pi}{3} \end{aligned}$$

There are eight values of x in $[0, 2\pi)$ which solve the equation.

3. Prove the identity $\sec 2u = \dfrac{\sec^2 u}{2 - \sec^2 u}$.

$$\sec 2u = \frac{1}{\cos 2u}$$

Pause to figure out the trig identity we need.

$$\begin{aligned} \cos(u - v) &= \cos u \cos v + \sin u \sin v \\ \cos(u + v) = \cos(u - (-v)) &= \cos u \cos(-v) + \sin u \sin(-v) \\ &= \cos u \cos v - \sin u \sin v \\ \cos(2u) &= \cos^2 u - \sin^2 u \end{aligned}$$

Back to our problem:

$$\begin{aligned}
\sec 2u &= \frac{1}{\cos 2u} \\
&= \frac{1}{\cos^2 u - \sin^2 u} \\
&= \frac{1}{\cos^2 u - \sin^2 u} \cdot \left(\frac{\sec^2 u}{\sec^2 u}\right) \\
&= \frac{\sec^2 u}{(\cos^2 u - \sin^2 u)\sec^2 u} \\
&= \frac{\sec^2 u}{(\cos^2 u - \sin^2 u)\frac{1}{\cos^2 u}} \\
&= \frac{\sec^2 u}{1 - \tan^2 u}
\end{aligned}$$

Pause to figure out the trig identity we need.

$$\begin{aligned}
\cos^2 x + \sin^2 x &= 1 \\
\frac{\cos^2 x}{\cos^2 x} + \frac{\sin^2 x}{\cos^2 x} &= \frac{1}{\cos^2 x} \\
1 + \tan^2 x &= \sec^2 x \\
\tan^2 x &= \sec^2 x - 1
\end{aligned}$$

Back to our problem:

$$\begin{aligned}
\sec 2u &= \frac{\sec^2 u}{1 - \tan^2 u} \\
&= \frac{\sec^2 u}{1 - (\sec^2 u - 1)} \\
\sec 2u &= \frac{\sec^2 u}{2 - \sec^2 u}
\end{aligned}$$

4. Solution is in notes for Sum and Difference Identities.