This is not a complete list of the types of problems to expect on the final exam.

Example Determine the domain of the function $f(x) = \sqrt{x - 12}$.

Since we cannot take the square root of a negative number and get a real number, the domain of f is all x such that $x - 12 \geq 0$, or $x \in [12, \infty)$.

Example Determine the domain of the function $f(x) = \dfrac{\sqrt{x}}{\ln x}$.

Since we cannot take the square root of a negative number and get a real number, we must have $x \in [0, \infty)$.

We also cannot have division by zero, so we must exclude $x = 1$, since $\ln 1 = 0$.

The domain of f is $x \in [0, 1) \cup (1, \infty)$.

Example Determine whether the function $g(x) = x^6 + x^2 + \sin x$ is even, odd, or neither. Use the algebraic technique to determine if a function is even or odd, rather than attempting to sketch the function.

$$\begin{aligned} g(-x) &= (-x)^6 + (-x)^2 + \sin(-x) \\ &= (-1)^6 x^6 + (-1)^2 x^2 + -\sin x \\ &= x^6 + x^2 - \sin x \end{aligned}$$

Since $g(-x) \neq g(x)$, and $g(-x) \neq -g(x)$, the function g is neither odd nor even.

Example Find a formula $f^{-1}(x)$ for the inverse of the function $f(x) = 4e^{3x-9}$ (you do not have to discuss domain and range).

$$\begin{aligned} y &= 4e^{3x-9} && \text{Step 1: let } y = f(x) \\ x &= 4e^{3y-9} && \text{Step 2: Flip } x \text{ and } y \\ \frac{x}{4} &= e^{3y-9} \\ \ln\left(\frac{x}{4}\right) &= \ln e^{3y-9} \\ \ln\left(\frac{x}{4}\right) &= 3y - 9 \\ \ln\left(\frac{x}{4}\right) + 9 &= 3y \\ \frac{\ln\left(\frac{x}{4}\right) + 9}{3} &= y \\ f^{-1}(x) &= \frac{\ln\left(\frac{x}{4}\right) + 9}{3} && \text{Finally, } f^{-1}(x) = y \end{aligned}$$

Example Write an equation for the linear function f that satisfies the conditions $f(-3) = -7$ and $f(5) = -11$.

The slope-intercept form for a straight line is $y = mx + b$, where m is the slope and b is the y-intercept.

$$\text{slope} = \frac{\text{rise}}{\text{run}} = \frac{-7-(-11)}{-3-5} = -\frac{1}{2}.$$

Therefore,

$$\begin{aligned} y &= -\frac{1}{2}x + b \\ -7 &= -\frac{1}{2}(-3) + b \text{ (substitute one of the points to determine } b) \\ b &= -7 - \frac{3}{2} = -\frac{14}{2} - \frac{3}{2} = -\frac{17}{2} \end{aligned}$$

The equation of the straight line through the two specified points is $y = -\frac{1}{2}x - \frac{17}{2}$.

Example Given the functions $f(x) = x^2 - 4$ and $g(x) = \sqrt{x} + 4$, determine the following compositions (simplify as much as possible). You do not have to discuss domains.

(a) $(f \circ f)(x)$

$$\begin{aligned} (f \circ f)(x) &= f(f(x)) \\ &= f(x^2-4) \\ &= (x^2-4)^2 - 4 \\ &= x^4 - 8x^2 + 16 - 4 \\ &= x^4 - 8x^2 + 12 \end{aligned}$$

(b) $(g \circ f)(x)$

$$\begin{aligned} (g \circ f)(x) &= g(f(x)) \\ &= g(x^2-4) \\ &= \sqrt{(x^2-4)} + 4 \\ &= \sqrt{x^2-4} + 4 \end{aligned}$$

Example For the quadratic function $f(x) = x^2 - 4x + 5$, convert to the vertex form $f(x) = a(x-h)^2 + k$ by completing the square.

$$\begin{aligned} f(x) &= x^2 - 4x + 5 \\ &= x^2 - 4x + (4-4) + 5 \\ &= (x^2 - 4x + 4) + (-4+5) \\ &= (x-2)^2 + 1 \end{aligned}$$

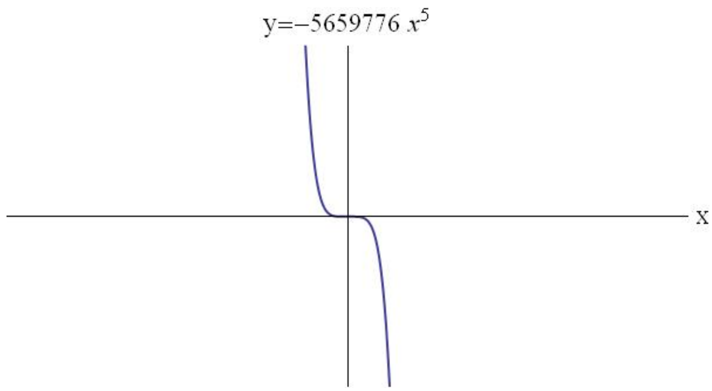
Example Given the function $g(x) = -(12x-7)^2(34x+89)^3$. State the degree of the polynomial, and the zeros with their multiplicity. Describe the end behaviour of this function, and determine $\lim_{x\to-\infty} g(x)$.

This is a polynomial of degree 5, with zeros $x = 7/12$ of multiplicity 2 and $x = -89/34$ of multiplicity 3.

For end behaviour, we look at the leading terms in each factor, since the leading terms will dominate for large $|x|$:

$$g(x) = -(12x-7)^2(34x+89)^3 \sim -(12x)^2(34x)^3 = -144 \cdot 39304x^5 = -5659776x^5.$$

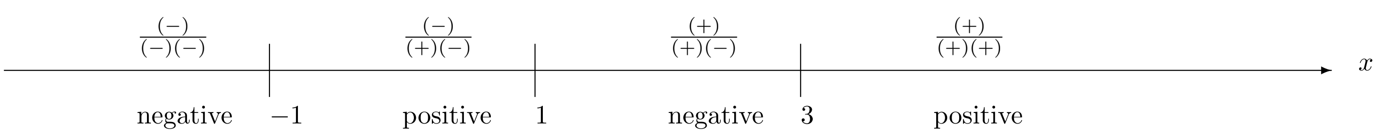
Therefore, we have end behaviour like the following for large $|x|$:



From the sketch, we see that $\lim_{x\to-\infty} g(x) = \infty$.

Example Solve the inequality $\dfrac{2(x-1)}{(x+1)(x-3)} \le 0$ using a sign chart.

The numerator is zero if $x = 1$, the denominator is zero if $x = -1, 3$. These are the possible values where the function will change sign.



From the sign diagram, we see that $\dfrac{1}{x+1} + \dfrac{1}{x-3} \le 0$ if $x \in (-\infty, -1) \cup [1, 3)$. We do not include $x = -1$ since the function is not defined there.

Example Given the function $f(x) = ax^2 + bx + c$, simplify the following expression as much as possible:

$$\frac{f(x_0+h) - f(x_0)}{h}$$

$$\begin{aligned}
\frac{f(x_0+h)-f(x_0)}{h} &= \frac{(a(x_0+h)^2+b(x_0+h)+c)-(ax_0^2+bx_0+c)}{h} \\
&= \frac{ax_0^2+ah^2+2ahx_0+bx_0+bh+c-ax_0^2-bx_0-c}{h} \\
&= \frac{ah^2+2ahx_0+bh}{h} \\
&= \frac{h(ah+2ax_0+b)}{h} \\
&= ah+2ax_0+b
\end{aligned}$$

Example Assuming x, y, and z are positive, use properties of logarithms to write the expression as a single logarithm.

$$\ln(xy)+2\ln(yz^2)-\ln(xz)$$

Solution:

$$\begin{aligned}
\ln(xy)+2\ln(yz^2)-\ln(xz) &= \ln(xy)+\ln((yz^2)^2)-\ln(xz) \\
&= \ln(xy)+\ln(y^2z^4)-\ln(xz) \\
&= \ln\left((xy)(y^2z^4)\right)-\ln(xz) \\
&= \ln\left(xy^3z^4\right)-\ln(xz) \\
&= \ln\left(\frac{xy^3z^4}{xz}\right) \\
&= \ln\left(y^3z^3\right)
\end{aligned}$$

Example Solve the equation $\dfrac{44}{1+4e^{-x/7}}=32$ algebraically.

Solution:

$$\begin{aligned}
\frac{44}{1+4e^{-x/7}} &= 32 \\
\frac{1+4e^{-x/7}}{44} &= \frac{1}{32} \\
1+4e^{-x/7} &= \frac{44}{32} \\
1+4e^{-x/7} &= \frac{11}{8} \\
4e^{-x/7} &= \frac{11}{8}-1 \\
4e^{-x/7} &= \frac{3}{8} \\
e^{-x/7} &= \frac{3}{32}
\end{aligned}$$

$$
\begin{aligned}
\ln e^{-x/7} &= \ln\left(\frac{3}{32}\right) \\
-\frac{x}{7} &= \ln\left(\frac{3}{32}\right) \\
x &= -7\ln\left(\frac{3}{32}\right) = -7\ln\left(\left(\frac{32}{3}\right)^{-1}\right) = 7\ln\left(\frac{32}{3}\right)
\end{aligned}
$$

Example Solve the equation $\ln x - \dfrac{1}{2}\ln(x+4) = 0$ algebraically. Be sure to eliminate any extraneous solutions.

$$
\begin{aligned}
\ln x - \frac{1}{2}\ln(x+4) &= 0 \\
\ln x - \ln\left((x+4)^{1/2}\right) &= 0 \\
\ln\left(\frac{x}{(x+4)^{1/2}}\right) &= 0 \\
e^{\ln\left(\frac{x}{(x+4)^{1/2}}\right)} &= e^0 \\
\frac{x}{(x+4)^{1/2}} &= 1 \\
x &= (x+4)^{1/2} \\
x^2 &= \left((x+4)^{1/2}\right)^2 \\
x^2 &= x+4 \\
x^2 - x - 4 &= 0 \\
x &= \frac{-b \pm \sqrt{b^2-4ac}}{2a} \\
&= \frac{1 \pm \sqrt{(-1)^2 - 4(1)(-4)}}{2} \\
&= \frac{1 \pm \sqrt{17}}{2}
\end{aligned}
$$

From the original equation, we must have $x + 10 > 0$ and $x > 0$, which are both satisfied if $x > 0$. These conditions are necessary for the logarithms to be defined.

The solution $\dfrac{1-\sqrt{17}}{2} < 0$, so it is an extraneous solution.

The only solution to the original equation is $\dfrac{1+\sqrt{17}}{2} > 0$.

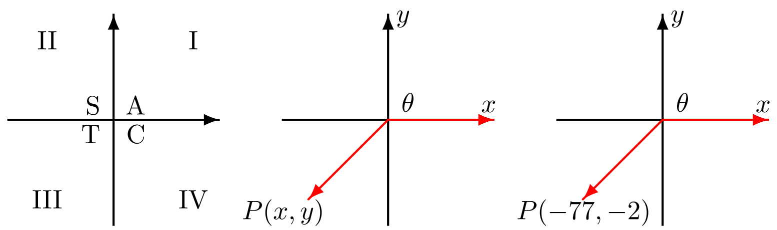
Example Given $f(x) = \frac{1}{2}\ln(x+2)$, $g(x) = e^x$. Find $(g \circ f)(x)$, and simplify as much as possible. Your final answer should **not** have exponentials and logarithms in them.

$$
(g \circ f)(x) = g(f(x))
$$

$$
\begin{aligned}
&= g(\frac{1}{2}\ln(x+2)) \\
&= g(\ln(\sqrt{x+2})) \\
&= e^{\ln(\sqrt{x+2})} \\
&= \sqrt{x+2}
\end{aligned}
$$

Example Find $\csc\theta$ if $\tan\theta = \dfrac{77}{2}$ and $\sin\theta < 0$.

Since $\sin\theta$ is less than zero, we must be in either Quadrant III or IV.
Since $\tan\theta$ is greater than zero we must be in either Quadrant I or III.
Therefore, the angle θ has a terminal side in Quadrant III.



Since $\tan\theta = \dfrac{y}{x} = \dfrac{77}{2} = \dfrac{-77}{-2}$, we have $y = -77$, $x = -2$.

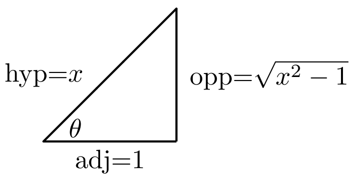
The distance $r = \sqrt{x^2+y^2} = \sqrt{(-77)^2+(-2)^2} = \sqrt{5933}$.

$$\text{Therefore, } \csc\theta = \frac{1}{\sin\theta} = \frac{r}{y} = \frac{\sqrt{5933}}{-77} = -\frac{\sqrt{5933}}{77}.$$

Example Find an algebraic expression equivalent to the expression $\sin\left(\arccos\left(\frac{1}{x}\right)\right)$.

To simplify this let $\theta = \arccos\left(\frac{1}{x}\right)$. This means $\cos\theta = \dfrac{1}{x} = \dfrac{\text{adj}}{\text{hyp}}$.

Construct a reference triangle



The length of the opposite side was found using the Pythagorean theorem

$$\sin\left(\arccos\left(\frac{1}{x}\right)\right) = \sin\theta = \frac{\text{opp}}{\text{hyp}} = \frac{\sqrt{x^2-1}}{x}.$$

Example Solve $\cos 2x + \cos x = 0$ algebraically for exact solutions in the interval $[0, 2\pi)$.

$$\begin{aligned}\cos 2x + \cos x &= \cos^2 x - \sin^2 x + \cos x \\ &= \cos^2 x - (1 - \cos^2 x) + \cos x \\ &= 2\cos^2 x + \cos x - 1 = 0\end{aligned}$$

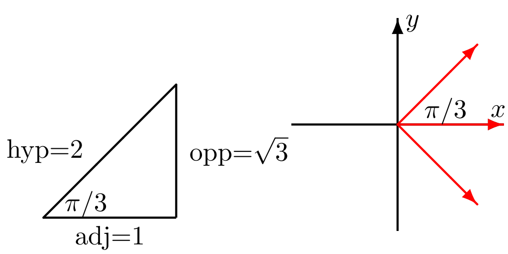
Let $y = \cos x$. Then

$$\begin{aligned}\cos 2x + \cos x &= 2\cos^2 x + \cos x - 1 = 0 \\ &= 2y^2 + y - 1 = 0 \\ y &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-1 \pm \sqrt{1+8}}{4} = \frac{-1 \pm 3}{4} = \frac{2}{4} \text{ or } \frac{-4}{4} = \frac{1}{2} \text{ or } -1\end{aligned}$$

So we must solve $y = \cos x = 1/2$ and $y = \cos x = -1$.

The equation $\cos x = -1$ has a solution of π in the interval $[0, 2\pi)$.

The equation $\cos x = \text{adj}/\text{hyp} = 1/2$ corresponds to one of our special triangles:



So the solution is $\pi/3$.

There is also a solution in Quadrant IV at $2\pi - \pi/3 = 5\pi/3$ in the interval $[0, 2\pi)$.

The solutions to $\cos 2x + \cos x = 0$ in the interval $[0, 2\pi)$ are $\dfrac{\pi}{3}, \pi, \dfrac{5\pi}{3}$.

Example Find the value of $\sin\left(\dfrac{\pi}{12}\right)$ exactly using an angle difference formula.

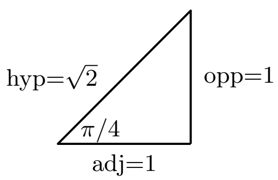
Solution:

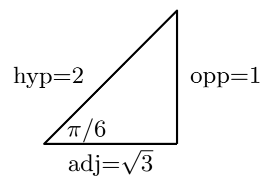
First, we need to figure out how to relate $\pi/12$ to some of our special angles, since we are told to find this answer exactly.

$$\frac{\pi}{12} = \frac{2\pi}{24} = \frac{6\pi - 4\pi}{24} = \frac{\pi}{4} - \frac{\pi}{6}.$$

Therefore,

$$
\begin{aligned}
\sin\left(\frac{\pi}{12}\right) &= \sin\left(\frac{\pi}{4}-\frac{\pi}{6}\right)\\
&= \sin\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{6}\right)-\cos\left(\frac{\pi}{4}\right)\sin\left(\frac{\pi}{6}\right), \quad \text{use } \sin(u-v)=\sin u\cos v-\cos u\sin v\\
&= \left(\frac{1}{\sqrt{2}}\right)\left(\frac{\sqrt{3}}{2}\right)-\left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{2}\right), \quad \text{using reference triangles below}\\
&= \frac{\sqrt{3}}{2\sqrt{2}}-\frac{1}{2\sqrt{2}}=\frac{\sqrt{3}-1}{2\sqrt{2}}
\end{aligned}
$$





Example Use the power reducing identities to prove the identity $\sin^4 x = \frac{1}{8}(3-4\cos 2x+\cos 4x)$.

Solution:

$$
\begin{aligned}
\sin^4 x &= \left(\sin^2 x\right)^2\\
&= \left(\frac{1-\cos 2x}{2}\right)^2, \quad \text{using } \sin^2 u=\frac{1-\cos 2u}{2}, \text{ with } u=x.\\
&= \frac{1}{4}\left(1-\cos 2x\right)^2\\
&= \frac{1}{4}\left(1+\cos^2 2x-2\cos 2x\right)\\
&= \frac{1}{4}\left(1+\left(\frac{1+\cos 4x}{2}\right)-2\cos 2x\right), \quad \text{using } \cos^2 u=\frac{1+\cos 2u}{2}, \text{ with } u=2x.\\
&= \frac{1}{4}\left(\frac{2}{2}+\frac{1+\cos 4x}{2}-\frac{4\cos 2x}{2}\right)\\
&= \frac{1}{8}\left(2+1+\cos 4x-4\cos 2x\right)\\
&= \frac{1}{8}\left(3+\cos 4x-4\cos 2x\right)\\
&= \frac{1}{8}\left(3-4\cos 2x+\cos 4x\right)
\end{aligned}
$$

Example Convert the rectangular equation $(x+3)^2+(y+3)^2=18$ to a polar equation.

We simply use our relations:

$$
\begin{aligned}
x &= r\cos\theta\\
y &= r\sin\theta,
\end{aligned}
$$

$$\begin{aligned}
(x+3)^2+(y+3)^2 &= 18\\
(r\cos\theta+3)^2+(r\sin\theta+3)^2 &= 18\\
(r^2\cos^2\theta+9+6r\cos\theta)+(r^2\sin^2\theta+9+6r\sin\theta) &= 18\\
r^2(\cos^2\theta+\sin^2\theta)+18+6r\cos\theta+6r\sin\theta &= 18\\
r^2(1)+6r\cos\theta+6r\sin\theta &= 18-18\\
r^2+6r\cos\theta+6r\sin\theta &= 0\\
r(r+6\cos\theta+6\sin\theta) &= 0\\
r+6\cos\theta+6\sin\theta &= 0, \qquad r\neq 0\\
r &= -6\cos\theta-6\sin\theta
\end{aligned}$$

Example Solve the system of equations

$$y^2 = -x+9 \qquad (1)$$
$$y = -x \qquad (2)$$

Substitute Eq. (2) into Eq. (1):

$$\begin{aligned}
y^2 &= -x+9\\
(-x)^2 &= -x+9\\
x^2+x-9 &= 0\\
x &= \frac{-b\pm\sqrt{b^2-4ac}}{2a}\\
&= \frac{-1\pm\sqrt{1+36}}{2}\\
&= \frac{-1\pm\sqrt{37}}{2}
\end{aligned}$$

For $x=\frac{-1+\sqrt{37}}{2}$:

$$y = -x = -\frac{-1+\sqrt{37}}{2} = \frac{1-\sqrt{37}}{2}$$

For $x=\frac{-1-\sqrt{37}}{2}$:

$$y = -x = -\frac{-1-\sqrt{37}}{2} = \frac{1+\sqrt{37}}{2}$$

The two solutions are $(x,y)=\left(\frac{-1+\sqrt{37}}{2},\frac{1-\sqrt{37}}{2}\right)$ and $(x,y)=\left(\frac{-1-\sqrt{37}}{2},\frac{1+\sqrt{37}}{2}\right)$

Example Starting from $\cos(u-v) = \cos u \cos v + \sin u \sin v$, derive an expression for $\sin(u+v)$.

$$\begin{aligned}
\cos(u-v) &= \cos u \cos v + \sin u \sin v \\
\sin(u+v) &= \cos\left(\frac{\pi}{2} - (u+v)\right) \\
&= \cos\left(\frac{\pi}{2} - u - v\right) \\
&= \cos\left(\left(\frac{\pi}{2} - u\right) - v\right) \\
&= \cos\left(\frac{\pi}{2} - u\right)\cos v + \sin\left(\frac{\pi}{2} - u\right)\sin v \\
&= \sin u \cos v + \cos u \sin v
\end{aligned}$$

Example Starting from $\cos(u+v) = \cos u \cos v - \sin u \sin v$, prove the identity $\cos^2 u = \dfrac{1+\cos 2u}{2}$.

$$\begin{aligned}
\cos(u+v) &= \cos u \cos v - \sin u \sin v \\
\cos(2u) = \cos(u+u) &= \cos u \cos u - \sin u \sin u \\
&= \cos^2 u - \sin^2 u \\
&= \cos^2 u - (1 - \cos^2 u) \\
\cos 2u &= 2\cos^2 u - 1 \\
\cos^2 u &= \frac{1+\cos 2u}{2}
\end{aligned}$$

Example Solve the system of equations

$$y^2 = x \tag{3}$$
$$x^2 = -8y \tag{4}$$

Substitute Eq. (3) into Eq. (4):

$$\begin{aligned}
x^2 &= -8y \\
(y^2)^2 &= -8y \\
y^4 &= -8y \\
y^4 + 8y &= 0 \\
y(y^3+8) &= 0 \\
y = 0 \quad &\text{or} \quad y^3 + 8 = 0 \\
& \quad\quad\;\; y^3 = -8 \\
& \quad\quad\;\; y = (-8)^{1/3} = -2
\end{aligned}$$

For $y = 0$:

$$x = y^2 = 0$$

For $y = -2$:

$$x = y^2 = (-2)^2 = 4$$

The two solutions are $(x, y) = (0, 0)$ and $(x, y) = (-2, 4)$

Example Sketch the ellipse $(x + 3)^2 + 16(y - 2)^2 = 4$.

First, get this in standard form:

$$\begin{aligned}
(x+3)^2 + 16(y-2)^2 &= 4 \\
\frac{(x+3)^2}{4} + 4(y-2)^2 &= 1 \\
\frac{(x+3)^2}{2^2} + \frac{(y-2)^2}{(1/2)^2} &= 1
\end{aligned}$$

The center of the ellipse is at $(-3, 2)$ and the "spread" in x is $a = 2$ and the "spread" in y is $b = 1/2$. This is enough to get the box the ellipse is inside, or you can do more work.

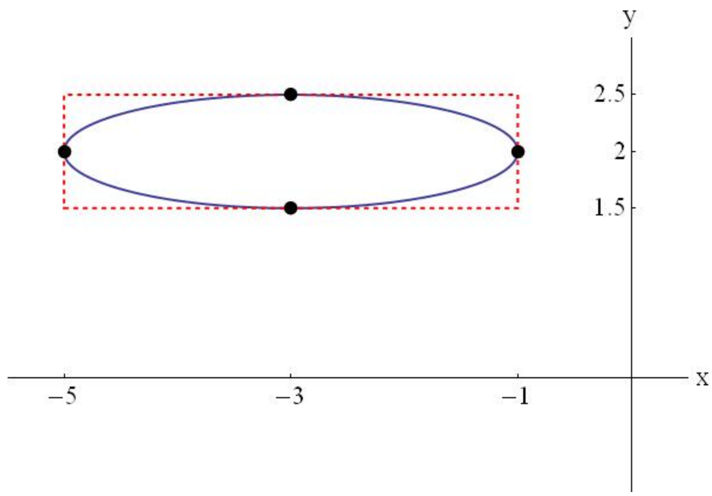
When $x = -3$, we have

$$\begin{aligned}
\frac{(0)^2}{2^2} + \frac{(y-2)^2}{(1/2)^2} &= 1 \\
y - 2 &= \pm\frac{1}{2} \\
y &= 2 \pm \frac{1}{2} = 1.5 \text{ or } 2.5
\end{aligned}$$

When $y = 2$, we have

$$\begin{aligned}
\frac{(x+3)^2}{2^2} + \frac{(0)^2}{(1/2)^2} &= 1 \\
x + 3 &= \pm 2 \\
x &= -3 \pm 2 = -1 \text{ or } -5
\end{aligned}$$

Here is a sketch. The four points we found above are included as the black dots, and helped us get the sketch.



For your Information (I won't ask you about foci on final exam):

The focal axis is $y = 2$. The center is $(03, 2)$

To get the foci we need $c = \pm\sqrt{4 - \frac{1}{4}} = \pm\frac{\sqrt{3}}{2}$.

The foci are $\left(-3 + \frac{\sqrt{3}}{2}, 2\right)$ and $\left(-3 - \frac{\sqrt{3}}{2}, 2\right)$. Here is a sketch that includes the foci as red dots: