Note: You can expect other types of questions on the test than the ones presented here!

The formulas I have memorized:

$\boxed{\cos^2 x + \sin^2 x = 1}$

$\boxed{\cos(u - v) = \cos u \cos v + \sin u \sin v}$

$\boxed{\sin(u + v) = \sin u \cos v + \cos u \sin v}$

Any other formula I need I will derive from these.

Questions

Example 1 Use the power reducing identities to prove the identity $\cos^4 x = \frac{1}{8}(3 + 4\cos 2x + \cos 4x)$.

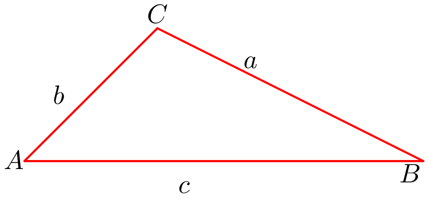
Example 2 Solve $\sin 2x + \sin 4x = 0$ exactly for all solutions in the interval $[0, 2\pi)$.

Example 3 Prove the identity $\sec 2u = \dfrac{\sec^2 u}{2 - \sec^2 u}$.

Example 4 Solve $\tan(x/2) = \sin x$ for $x \in [0, \pi)$.

Example 5 Show why $\tan\left(-\frac{\pi}{12}\right) = -2 + \sqrt{3}$ using angle difference formulas.

Problem 6 Derive the Law of Cosines, $a^2 = b^2 + c^2 - 2bc\cos A$, given the triangle



Solutions

Example 1 Use the power reducing identities to prove the identity $\cos^4 x = \dfrac{1}{8}(3 + 4\cos 2x + \cos 4x)$.

$$\cos^4 x = \left(\cos^2 x\right)^2$$

Pause to figure out the trig identity we need. It looks like we want a power reducing identity, since we are headed towards something with no powers of trig functions.

$$\begin{aligned}
\cos(u-v) &= \cos u \cos v + \sin u \sin v \\
\cos(u+v) = \cos(u-(-v)) &= \cos u \cos(-v) + \sin u \sin(-v) \\
&= \cos u \cos v - \sin u \sin v \\
\cos(2u) &= \cos^2 u - \sin^2 u \\
&= \cos^2 u - (1 - \cos^2 u) \\
&= 2\cos^2 u - 1 \\
\cos^2 u &= \frac{1+\cos 2u}{2}
\end{aligned}$$

Back to our problem:

$$\begin{aligned}
\cos^4 x &= \left(\cos^2 x\right)^2 \\
&= \left(\frac{1+\cos 2x}{2}\right)^2, \quad \text{using } \cos^2 u = \frac{1+\cos 2u}{2}, \text{ with } u = x. \\
&= \frac{1}{4}\left(1+\cos 2x\right)^2 \\
&= \frac{1}{4}\left(1+\cos^2 2x + 2\cos 2x\right) \\
&= \frac{1}{4}\left(1+\left(\frac{1+\cos 4x}{2}\right) + 2\cos 2x\right), \quad \text{using } \cos^2 u = \frac{1+\cos 2u}{2}, \text{ with } u = 2x. \\
&= \frac{1}{4}\left(\frac{2}{2} + \frac{1+\cos 4x}{2} + \frac{4\cos 2x}{2}\right) \\
&= \frac{1}{8}\left(2+1+\cos 4x + 4\cos 2x\right) \\
&= \frac{1}{8}\left(3+\cos 4x + 4\cos 2x\right) \\
&= \frac{1}{8}\left(3+4\cos 2x + \cos 4x\right)
\end{aligned}$$

Example 2 Solve $\sin 2x + \sin 4x = 0$ exactly for all solutions in the interval $[0, 2\pi)$.

$$\begin{aligned}
\sin 2x + \sin 4x &= \sin 2x + 2\sin 2x \cos 2x, \qquad \text{use } \sin 2u = 2\sin u \cos u \text{ with } u = 2x. \\
&= \sin 2x(1+2\cos 2x) = 0 \\
& \quad \sin 2x = 0 \quad \text{or} \quad 1 + 2\cos 2x = 0 \\
& \quad \sin y = 0 \quad \text{or} \quad 1 + 2\cos y = 0
\end{aligned}$$

Where we have let $y = 2x$. Since we want $x \in [0, 2\pi)$, we should search for all solutions $y \in [0, 4\pi)$.

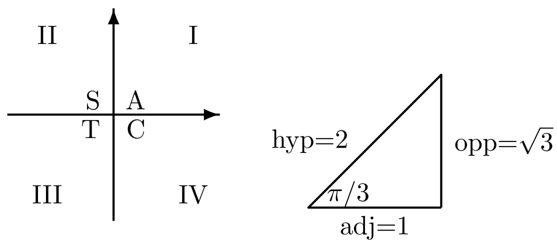
First, $\sin y = 0$ if $y = 0, \pi, 2\pi, 3\pi$. These are all the solutions for $y \in [0, 4\pi)$.

$$\begin{aligned} y = 2x = 0 &\quad\longrightarrow\quad x = 0 \\ y = 2x = \pi &\quad\longrightarrow\quad x = \frac{\pi}{2} \\ y = 2x = 2\pi &\quad\longrightarrow\quad x = \pi \\ y = 2x = 3\pi &\quad\longrightarrow\quad x = \frac{3\pi}{2} \end{aligned}$$

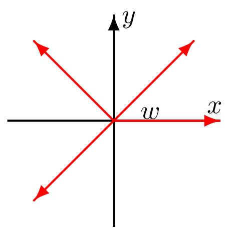
Now, $1 + 2\cos y = 0$, which means $\cos y = -\frac{1}{2}$.

Since the cosine is negative, we must be in either Quadrant II or III.

Let's figure out what the solution to $\cos w = \text{adj}/\text{hyp} = \frac{1}{2}$ is. This comes from one of our special triangles:



So $w = \pi/3$. We want the corresponding solutions in Quadrants II and III.



$$\begin{aligned} y &= \pi - w = \frac{2\pi}{3} \\ y &= \pi + w = \frac{4\pi}{3} \end{aligned}$$

Now we need all the solutions $y \in [0, 4\pi)$:

$$\begin{aligned} y &= \frac{2\pi}{3} \\ y &= \frac{4\pi}{3} \end{aligned}$$

$$\begin{aligned} y &= \frac{2\pi}{3} + 2\pi = \frac{8\pi}{3} \\ y &= \frac{4\pi}{3} + 2\pi = \frac{10\pi}{3} \end{aligned}$$

Now we need the solutions we seek, x:

$$\begin{aligned} y = 2x = \frac{2\pi}{3} &\longrightarrow x = \frac{\pi}{3} \\ y = 2x = \frac{4\pi}{3} &\longrightarrow x = \frac{2\pi}{3} \\ y = 2x = \frac{8\pi}{3} &\longrightarrow x = \frac{4\pi}{3} \\ y = 2x = \frac{10\pi}{3} &\longrightarrow x = \frac{5\pi}{3} \end{aligned}$$

There are eight values of x in $[0, 2\pi)$ which solve the equation.

Example 3 Prove the identity $\sec 2u = \dfrac{\sec^2 u}{2 - \sec^2 u}$.

$$\sec 2u = \frac{1}{\cos 2u}$$

Pause to figure out the trig identity we need.

$$\begin{aligned} \cos(u - v) &= \cos u \cos v + \sin u \sin v \\ \cos(u + v) = \cos(u - (-v)) &= \cos u \cos(-v) + \sin u \sin(-v) \\ &= \cos u \cos v - \sin u \sin v \\ \cos(2u) &= \cos^2 u - \sin^2 u \end{aligned}$$

Back to our problem:

$$\begin{aligned} \sec 2u &= \frac{1}{\cos 2u} \\ &= \frac{1}{\cos^2 u - \sin^2 u} \\ &= \frac{1}{\cos^2 u - \sin^2 u} \cdot \left(\frac{\sec^2 u}{\sec^2 u}\right) \\ &= \frac{\sec^2 u}{(\cos^2 u - \sin^2 u)\sec^2 u} \\ &= \frac{\sec^2 u}{(\cos^2 u - \sin^2 u)\frac{1}{\cos^2 u}} \\ &= \frac{\sec^2 u}{1 - \tan^2 u} \end{aligned}$$

Pause to figure out the trig identity we need.

$$\cos^2 x + \sin^2 x = 1$$

$$\begin{aligned}
\frac{\cos^2 x}{\cos^2 x} + \frac{\sin^2 x}{\cos^2 x} &= \frac{1}{\cos^2 x} \\
1 + \tan^2 x &= \sec^2 x \\
\tan^2 x &= \sec^2 x - 1
\end{aligned}$$

Back to our problem:

$$\begin{aligned}
\sec 2u &= \frac{\sec^2 u}{1 - \tan^2 u} \\
&= \frac{\sec^2 u}{1 - (\sec^2 u - 1)} \\
\sec 2u &= \frac{\sec^2 u}{2 - \sec^2 u}
\end{aligned}$$

Example 4 Solve $\tan(x/2) = \sin x$ for $x \in [0, \pi)$.

We need to convert the half angle tangent function to trig functions of x.

Pause to work out some trig identities:

$$\begin{aligned}
\cos(u - v) &= \cos u \cos v + \sin u \sin v \\
\cos(u + v) = \cos(u - (-v)) &= \cos u \cos(-v) + \sin u \sin(-v) \\
&= \cos u \cos v - \sin u \sin v \\
\cos(2u) &= \cos^2 u - \sin^2 u \\
&= \cos^2 u - (1 - \cos^2 u) \\
&= 2\cos^2 u - 1 \\
\cos^2 u &= \frac{1}{2}(1 + \cos 2u)
\end{aligned}$$

$$\begin{aligned}
\cos(2u) &= \cos^2 u - \sin^2 u \\
&= (1 - \sin^2 u) - 1 \\
&= 1 - 2\sin^2 u \\
\sin^2 u &= \frac{1}{2}(1 - \cos 2u)
\end{aligned}$$

$$\begin{aligned}
\tan^2 u &= \frac{\sin^2 u}{\cos^2 u} \\
&= \frac{1 - \cos 2u}{1 + \cos 2u} \\
&= \frac{1 - \cos 2u}{1 + \cos 2u} \cdot \left(\frac{1 - \cos 2u}{1 - \cos 2u}\right) \\
&= \frac{(1 - \cos 2u)^2}{1 - \cos^2 2u} \\
&= \frac{(1 - \cos 2u)^2}{\sin^2 2u} \\
\tan^2 u &= \left(\frac{1 - \cos 2u}{\sin 2u}\right)^2
\end{aligned}$$

$$\tan u = \frac{1-\cos 2u}{\sin 2u}$$

The last line is true since $\sin 2u$ and $\tan u$ have the same sign at any point.

This was a serious amount of work, but look at how many trig identities we found along the way! On a test, these identities can be reused in other problems if needed. This is probably the most work you would ever have to do so derive certain trig identities; most of the time the derivation will be significantly shorter.

Now we can work on our problem:

$$\begin{aligned}
\tan(x/2) &= \sin x \\
\frac{1-\cos x}{\sin x} &= \sin x, \quad \text{(above formula with } u = x/2) \\
1-\cos x &= \sin^2 x \\
1-\cos x &= 1-\cos^2 x \\
-\cos x &= -\cos^2 x \\
\cos^2 x - \cos x &= 0 \\
\cos x(\cos x - 1) &= 0
\end{aligned}$$

So we need to solve $\cos x = 0$ and $\cos x - 1 = 0$.

For the first, $\cos x = 0$ for $x = \pi/2 \in [0, \pi)$.
For the second, $\cos x = 1$ for $x = 0 \in [0, \pi)$.

The two solutions are $x = 0, \frac{\pi}{2}$ for $x \in [0, \pi)$.

Example 5 Show why $\tan\left(-\frac{\pi}{12}\right) = -2+\sqrt{3}$ using angle difference formulas.

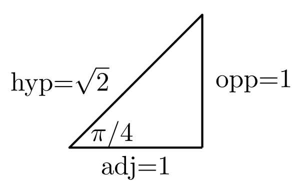
We can write the tangent in terms of sine and cosine functions:

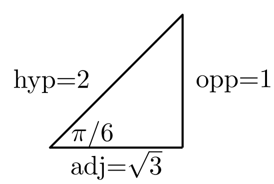
$$\tan\left(-\frac{\pi}{12}\right) = \frac{\sin\left(-\frac{\pi}{12}\right)}{\cos\left(-\frac{\pi}{12}\right)}.$$

Now, we need to figure out how to relate $-\pi/12$ to some of our special angles, since we are told to find this answer exactly.

$$\frac{-\pi}{12} = \frac{-2\pi}{24} = \frac{4\pi - 6\pi}{24} = \frac{\pi}{6} - \frac{\pi}{4}.$$

Here are the reference triangles we will need:





We need cosine and sine of a difference identities, which are

$$\begin{aligned}
\cos(u-v) &= \cos u\cos v + \sin u \sin v \text{ (memorized)}\\
\sin(u+v) &= \sin u\cos v + \cos u \sin v \text{ (memorized)}\\
\sin(u-v) &= \sin(u+(-v)) \text{ (work this out, using above identity)}\\
&= \sin u\cos(-v) + \cos u \sin(-v)\\
\sin(u-v) &= \sin u\cos v - \cos u \sin v \text{ (since cosine is even and sine is odd)}
\end{aligned}$$

We have what we need to solve the problem.

Therefore,

$$\begin{aligned}
\sin\left(-\frac{\pi}{12}\right) &= \sin\left(\frac{\pi}{6}-\frac{\pi}{4}\right)\\
&= \sin\left(\frac{\pi}{6}\right)\cos\left(\frac{\pi}{4}\right)-\cos\left(\frac{\pi}{6}\right)\sin\left(\frac{\pi}{4}\right), \quad \text{use } \sin(u-v)=\sin u\cos v-\cos u\sin v\\
&= \left(\frac{1}{2}\right)\left(\frac{1}{\sqrt{2}}\right)-\left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{\sqrt{2}}\right), \quad \text{using reference triangles above}\\
&= \frac{1}{2\sqrt{2}}-\frac{\sqrt{3}}{2\sqrt{2}}=\frac{1-\sqrt{3}}{2\sqrt{2}}
\end{aligned}$$

and

$$\begin{aligned}
\cos\left(-\frac{\pi}{12}\right) &= \cos\left(\frac{\pi}{6}-\frac{\pi}{4}\right)\\
&= \cos\left(\frac{\pi}{6}\right)\cos\left(\frac{\pi}{4}\right)+\sin\left(\frac{\pi}{6}\right)\sin\left(\frac{\pi}{4}\right), \quad \text{use } \cos(u-v)=\cos u\cos v+\sin u\sin v\\
&= \left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{\sqrt{2}}\right)+\left(\frac{1}{2}\right)\left(\frac{1}{\sqrt{2}}\right), \quad \text{using reference triangles above}\\
&= \frac{\sqrt{3}}{2\sqrt{2}}+\frac{1}{2\sqrt{2}}=\frac{\sqrt{3}+1}{2\sqrt{2}}
\end{aligned}$$

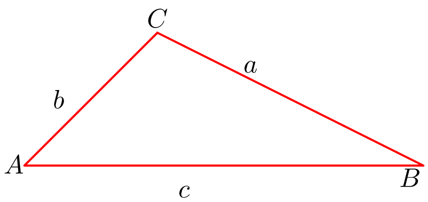
So we have

$$\tan\left(-\frac{\pi}{12}\right)=\frac{\sin\left(-\frac{\pi}{12}\right)}{\cos\left(-\frac{\pi}{12}\right)}=\left(\frac{1-\sqrt{3}}{2\sqrt{2}}\right)\times\left(\frac{2\sqrt{2}}{\sqrt{3}+1}\right)=\frac{1-\sqrt{3}}{1+\sqrt{3}}.$$

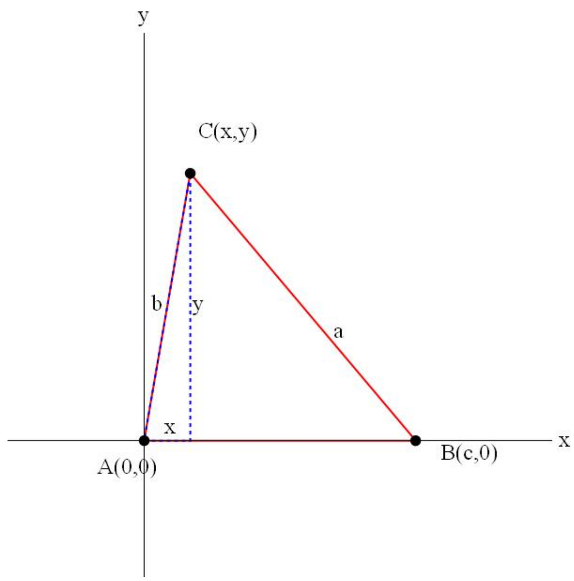
To get the final result asked for, we can rationalize the denominator:

$$\tan\left(-\frac{\pi}{12}\right)=\frac{1-\sqrt{3}}{1+\sqrt{3}}=\frac{1-\sqrt{3}}{1+\sqrt{3}}\times\left(\frac{1-\sqrt{3}}{1-\sqrt{3}}\right)=\frac{1-2\sqrt{3}+3}{1-3}=\frac{4-2\sqrt{3}}{-2}=-2+\sqrt{3}.$$

Problem 6 Derive the Law of Cosines, $a^2 = b^2 + c^2 - 2bc\cos A$, given the triangle



The law of cosines is a generalization of the Pythagorean theorem. It can be derived in a manner similar to how we derived the formula for $\cos(u - v)$. Let's introduce a coordinate system (my triangle has changed in scale, but otherwise the edges a, b, and c all line up!):



The coordinates of the point C satisfy:

$$\frac{x}{b} = \cos A \quad \text{and} \quad \frac{y}{b} = \sin A$$

Therefore, $x = b\cos A$ and $y = b\sin A$. Using the distance formula, we can write for the distance from point C to B:

$$\begin{aligned}
a &= \sqrt{(x-c)^2 + (y-0)^2} \\
a^2 &= (x-c)^2 + y^2 \\
a^2 &= (b\cos A - c)^2 + (b\sin A)^2 \\
a^2 &= b^2\cos^2 A + c^2 - 2bc\cos A + b^2\sin^2 A \\
a^2 &= b^2(\cos^2 A + \sin^2 A) + c^2 - 2bc\cos A \\
a^2 &= b^2(1) + c^2 - 2bc\cos A \\
a^2 &= b^2 + c^2 - 2bc\cos A
\end{aligned}$$