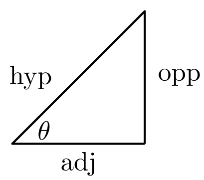
1011 Precalculus Chapter 5: Concepts to Review

Chapter 4

You will need to have proficiency with Chapter 4 to understand Chapter 5. The more important things from Chapter 4 are:

- angular measure, degree, radians (4.1)
- right triangle trigonometry, Pythagorean theorem (acute angles) (4.2)



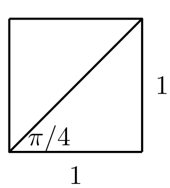
$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} \quad \csc\theta = \frac{\text{hypotenuse}}{\text{opposite}}$$
$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} \quad \sec\theta = \frac{\text{hypotenuse}}{\text{adjacent}}$$
$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}} \quad \cot\theta = \frac{\text{adjacent}}{\text{opposite}}$$

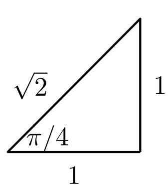
- special triangles (4.2)

  The six basic trigonometric functions relate the angle θ to ratios of the length of the sides of the right triangle. For certain triangles, the trig functions of the angles can be found geometrically. These special triangles occur frequently enough that it is expected that you learn the value of the trig functions for the special angles.

A 45-45-90 Triangle

Consider the square given below.





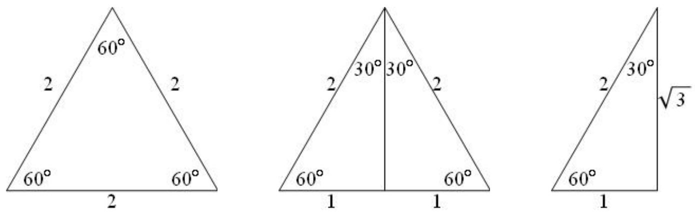
The angle here must be $\pi/4$ radians, since this triangle is half of a square of side length 1.

Now, we can write down all the trig functions for an angle of $\pi/4$ radians $= 45$ degrees:

$$\sin\left(\frac{\pi}{4}\right) = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{1}{\sqrt{2}} \qquad \csc\left(\frac{\pi}{4}\right) = \frac{1}{\sin\left(\frac{\pi}{4}\right)} = \sqrt{2}$$
$$\cos\left(\frac{\pi}{4}\right) = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{1}{\sqrt{2}} \qquad \sec\left(\frac{\pi}{4}\right) = \frac{1}{\cos\left(\frac{\pi}{4}\right)} = \sqrt{2}$$
$$\tan\left(\frac{\pi}{4}\right) = \frac{\text{opposite}}{\text{adjacent}} = \frac{1}{1} = 1 \qquad \cot\left(\frac{\pi}{4}\right) = \frac{1}{\tan\left(\frac{\pi}{4}\right)} = 1$$

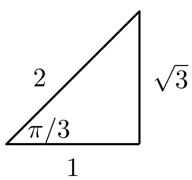
A 30-60-90 Triangle

Consider the equilateral triangle given below. Geometry allows us to construct a 30-60-90 triangle:



We can now determine the six trigonometric functions at two more angles!

$60^{\text{o}} = \dfrac{\pi}{3}$ radians:

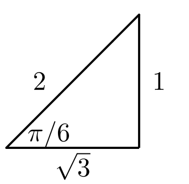


$$\sin\left(\frac{\pi}{3}\right) = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{\sqrt{3}}{2} \qquad \csc\left(\frac{\pi}{3}\right) = \frac{1}{\sin\left(\frac{\pi}{3}\right)} = \frac{2}{\sqrt{3}}$$

$$\cos\left(\frac{\pi}{3}\right) = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{1}{2} \qquad \sec\left(\frac{\pi}{3}\right) = \frac{1}{\cos\left(\frac{\pi}{3}\right)} = 2$$

$$\tan\left(\frac{\pi}{3}\right) = \frac{\text{opposite}}{\text{adjacent}} = \frac{\sqrt{3}}{1} = \sqrt{3} \qquad \cot\left(\frac{\pi}{3}\right) = \frac{1}{\tan\left(\frac{\pi}{3}\right)} = \frac{1}{\sqrt{3}}$$

$30^{\text{o}} = \dfrac{\pi}{6}$ radians:

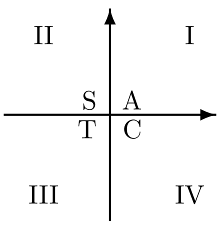


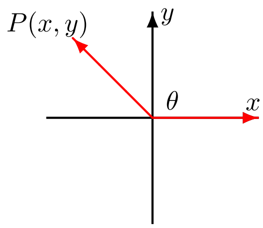
$$\sin\left(\frac{\pi}{6}\right) = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{1}{2} \qquad \csc\left(\frac{\pi}{6}\right) = \frac{1}{\sin\left(\frac{\pi}{3}\right)} = 2$$

$$\cos\left(\frac{\pi}{6}\right) = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{\sqrt{3}}{2} \qquad \sec\left(\frac{\pi}{6}\right) = \frac{1}{\cos\left(\frac{\pi}{3}\right)} = \frac{2}{\sqrt{3}}$$

$$\tan\left(\frac{\pi}{6}\right) = \frac{\text{opposite}}{\text{adjacent}} = \frac{1}{\sqrt{3}} \qquad \cot\left(\frac{\pi}{6}\right) = \frac{1}{\tan\left(\frac{\pi}{3}\right)} = \frac{\sqrt{3}}{1} = \sqrt{3}$$

- initial side, terminal side, vertex, standard position, coterminal angles, quadrantal angles, quadrants, CAST (4.3)





If we label the point at the end of the terminal side as $P(x, y)$, and if we let $r = \sqrt{x^2 + y^2}$, we can construct the following relationships between the six trig functions and our diagram:

$$\cos\theta = \frac{x}{r}, \qquad \sin\theta = \frac{y}{r}, \qquad \tan\theta = \frac{y}{x},\ x \neq 0$$
$$\csc\theta = \frac{r}{y},\ y \neq 0, \qquad \sec\theta = \frac{r}{x},\ x \neq 0, \qquad \cot\theta = \frac{x}{y},\ y \neq 0$$

Chapter 5

You will need to be able to know the basic trig identities, or derive them. I recommend memorizing a few, and deriving others that you will need when necessary. I would memorize $\cos^2 x + \sin^2 x = 1$, $\cos(u - v) = \cos u \cos v + \sin u \sin v$, $\sin(u + v) = \sin u \cos v + \cos u \sin v$.

- Basic Identities (5.1)

  From the definition of the trig functions:

$$\csc\theta = \frac{1}{\sin\theta} \qquad \sec\theta = \frac{1}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

$$\sin\theta = \frac{1}{\csc\theta} \qquad \cos\theta = \frac{1}{\sec\theta} \qquad \tan\theta = \frac{1}{\cot\theta}$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

- Pythagorean Identities (5.1)

$$\cos^2\theta + \sin^2\theta = 1$$

  Divide by $\cos^2\theta$:

$$\begin{aligned} \frac{\cos^2\theta}{\cos^2\theta} + \frac{\sin^2\theta}{\cos^2\theta} &= \frac{1}{\cos^2\theta} \\ 1 + \tan^2\theta &= \sec^2\theta \end{aligned}$$

  Divide by $\sin^2\theta$:

$$\begin{aligned} \frac{\cos^2\theta}{\sin^2\theta} + \frac{\sin^2\theta}{\sin^2\theta} &= \frac{1}{\sin^2\theta} \\ \cot^2\theta + 1 &= \csc^2\theta \end{aligned}$$

- Cofunction Identities (5.1)

$$\sin\theta = \cos\left(\frac{\pi}{2} - \theta\right) \qquad \cos\theta = \sin\left(\frac{\pi}{2} - \theta\right) \qquad \tan\theta = \cot\left(\frac{\pi}{2} - \theta\right)$$

$$\csc\theta = \sec\left(\frac{\pi}{2} - \theta\right) \qquad \sec\theta = \csc\left(\frac{\pi}{2} - \theta\right) \qquad \cot\theta = \tan\left(\frac{\pi}{2} - \theta\right)$$

- Even/Odd Identities (5.1)

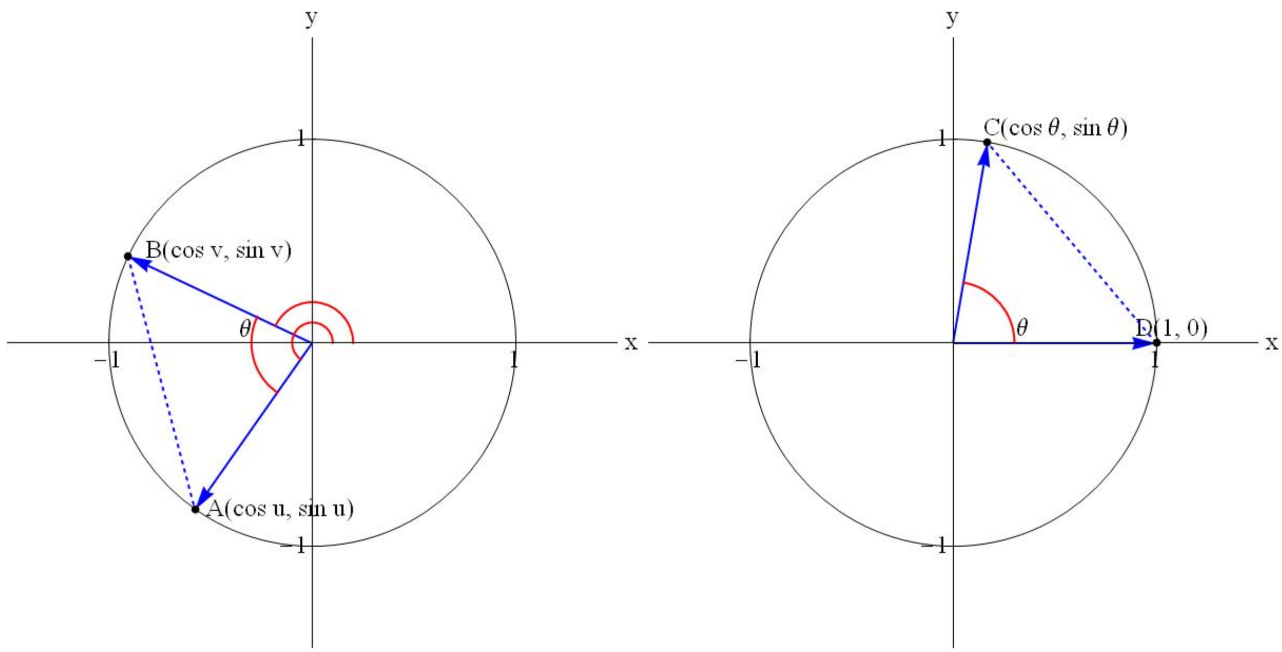
$$\sin(-\theta) = -\sin\theta \qquad \cos(-\theta) = \cos\theta \qquad \tan(-\theta) = -\tan\theta$$

$$\csc(-\theta) = -\csc\theta \qquad \sec(-\theta) = \sec\theta \qquad \cot(-\theta) = -\cot\theta$$

- The Cosine of a Difference Identity Derivation (5.3)

  To get the cosine of a difference, let's draw a diagram involving the unit circle and see what we can learn.

  The angle u leads to a point $A(\cos u, \sin u)$ on the unit circle.
  The angle v leads to a point $B(\cos v, \sin v)$ on the unit circle.
  The angle $\theta = u - v$ is the angle between the the terminal sides of u and v.
  The dotted line connects the points A and B.



We can rotate the geometry of this picture so that the angle θ is in standard position.

The dashed lines are the same length in both pictures. Therefore, we can use the distance between two points formula

$$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \quad \text{(see page 16)}$$

and we can write:

$$\sqrt{(\cos u - \cos v)^2 + (\sin u - \sin v)^2} = \sqrt{(\cos\theta - 1)^2 + (\sin\theta - 0)^2}$$

Now all we have to do is simplify this expression! Remember, $\theta = u - v$, so we want to solve this for $\cos\theta = \cos(u-v)$.

$$\begin{aligned}
(\sqrt{(\cos u - \cos v)^2 + (\sin u - \sin v)^2})^2 &= (\sqrt{(\cos\theta - 1)^2 + (\sin\theta - 0)^2})^2 \\
(\cos u - \cos v)^2 + (\sin u - \sin v)^2 &= (\cos\theta - 1)^2 + (\sin\theta - 0)^2 \\
(\cos^2 u + \cos^2 v - 2\cos u\cos v) + (\sin^2 u + \sin^2 v - 2\sin u\sin v) &= (\cos^2\theta + 1 - 2\cos\theta) + \sin^2\theta \\
(\cos^2 u + \sin^2 u) - 2\cos u\cos v + (\cos^2 v + \sin^2 v) - 2\sin u\sin v &= (\cos^2\theta + \sin^2\theta) + 1 - 2\cos\theta) \\
(1) - 2\cos u\cos v + (1) - 2\sin u\sin v &= (1) + 1 - 2\cos\theta \\
2 - 2\cos u\cos v - 2\sin u\sin v &= 2 - 2\cos\theta \\
2 - 2\cos u\cos v - 2\sin u\sin v &= 2 - 2\cos\theta \\
-2\cos u\cos v - 2\sin u\sin v &= -2\cos\theta \\
+\cos u\cos v + \sin u\sin v &= +\cos\theta \\
\cos\theta = \cos(u-v) &= \cos u\cos v + \sin u\sin v
\end{aligned}$$

We have arrived at the trig identity $\cos(u-v) = \cos u\cos v + \sin u\sin v$.

- The Cosine of a Sum Identity (5.3) $\cos(u+v) = \cos u\cos v - \sin u\sin v$.
- The Sine of a Sum/Difference Identities (5.3) $\sin(u \pm v) = \sin u\cos v \pm \cos u\sin v$.
- The Tangent of a Difference or Sum Identities (5.3) $\tan(u \pm v) = \dfrac{\sin(u \pm v)}{\cos(u \pm v)} = \dfrac{\sin u\cos v \pm \cos u\sin v}{\cos u\cos v \mp \sin u\sin v}$.
- Double Angle Identities (5.4)

  The double angle identities are found from letting $u = v$ in the sum identities.

$$\begin{aligned}
\cos(u+v) &= \cos u\cos v - \sin u\sin v \\
\cos(2u) = \cos(u+u) &= \cos u\cos u - \sin u\sin u \\
&= \cos^2 u - \sin^2 u \\
&= \cos^2 u - (1 - \cos^2 u) \\
&= 2\cos^2 u - 1 \\
&= 2(1 - \sin^2 u) - 1 \\
&= 1 - 2\sin^2 u
\end{aligned}$$

$$\begin{aligned}
\sin(u+v) &= \sin u\cos v + \cos u\sin v \\
\sin(2u) = \sin(u+v) &= \sin u\cos u + \cos u\sin u \\
&= 2\sin u\cos u
\end{aligned}$$

$$\begin{aligned}
\tan(2u) = \frac{\sin(2u)}{\cos(2u)} &= \frac{2\sin u\cos u}{\cos^2 u - \sin^2 u} \\
&= \frac{2\sin u\cos u}{\cos^2 u - \sin^2 u} \cdot (1) \\
&= \frac{2\sin u\cos u}{\cos^2 u - \sin^2 u} \cdot \left(\frac{\left(\frac{1}{\cos^2 u}\right)}{\left(\frac{1}{\cos^2 u}\right)}\right) \\
&= \frac{2\frac{\sin u}{\cos u}}{\frac{\cos^2 u}{\cos^2 u} - \frac{\sin^2 u}{\cos^2 u}} \\
&= \frac{2\tan u}{1 - \tan^2 u}
\end{aligned}$$

- Power Reducing Identities (5.4)

  The power reducing identities are found by rearranging the double angle identities.

$$\begin{aligned} \cos(2u) &= 2\cos^2 u - 1 \\ \cos^2 u &= \frac{1+\cos 2u}{2} \end{aligned}$$

$$\begin{aligned} \cos(2u) &= 1 - 2\sin^2 u \\ \sin^2 u &= \frac{1-\cos 2u}{2} \end{aligned}$$

$$\begin{aligned} \tan^2 u = \frac{\sin^2 u}{\cos^2 u} &= \frac{\left(\frac{1-\cos 2u}{2}\right)}{\left(\frac{1+\cos 2u}{2}\right)} \\ &= \frac{\left(\frac{1-\cos 2u}{2}\right)}{\left(\frac{1+\cos 2u}{2}\right)} \cdot \left(\frac{2}{2}\right) \\ &= \frac{1-\cos 2u}{1+\cos 2u} \end{aligned}$$

- Half Angle Identities (5.4)

  The half angle identities are found from the power reducing identities. They have an inherent ambiguity in the sign of the square root, and this ambiguity can only be removed by checking which quadrant $u/2$ lies in on a case-by-case basis.

$$\begin{aligned} \cos^2 u &= \frac{1+\cos 2u}{2} \\ \cos^2(u/2) &= \frac{1+\cos u}{2} \\ \cos(u/2) &= \pm\sqrt{\frac{1+\cos u}{2}} \end{aligned}$$

$$\begin{aligned} \sin^2 u &= \frac{1-\cos 2u}{2} \\ \sin^2(u/2) &= \frac{1-\cos u}{2} \\ \sin(u/2) &= \pm\sqrt{\frac{1-\cos u}{2}} \end{aligned}$$

$$\begin{aligned} \tan^2 u &= \frac{1-\cos 2u}{1+\cos 2u} \\ \tan^2(u/2) &= \frac{1-\cos u}{1+\cos u} \\ \tan(u/2) &= \pm\sqrt{\frac{1-\cos u}{1+\cos u}} \end{aligned}$$

  For the half angle tangent identities, we can write two additional identities that do not have the ambiguity of the sign of the square root since the $\sin x$ and $\tan(x/2)$ are both negative in the same intervals.

$$\begin{aligned} \tan(u/2) &= \pm\sqrt{\frac{1-\cos u}{1+\cos u}} \\ &= \pm\sqrt{\frac{(1-\cos u)(1-\cos u)}{(1+\cos u)(1-\cos u)}} \end{aligned}$$

$$
\begin{aligned}
&= \pm\sqrt{\frac{(1-\cos u)^2}{(1-\cos^2 u)}} \\
&= \pm\sqrt{\frac{(1-\cos u)^2}{\sin^2 u}} = \frac{1-\cos u}{\sin u} \\
\tan(u/2) &= \frac{1-\cos u}{\sin u}\cdot\left(\frac{1+\cos u}{1+\cos u}\right) \\
&= \frac{(1-\cos u)(1+\cos u)}{\sin u(1+\cos u)} \\
&= \frac{1-\cos^2 u}{\sin u(1+\cos u)} = \frac{\sin^2 u}{\sin u(1+\cos u)} = \frac{\sin u}{1+\cos u}
\end{aligned}
$$

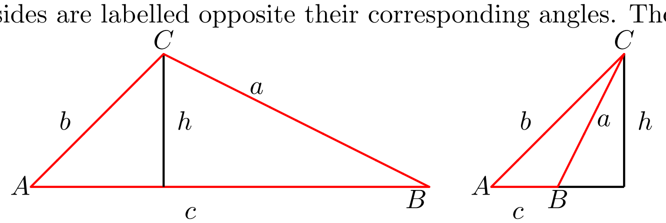
- Proving trig identities (5.2)

  Proof strategies:

  - The proof begins with the expression on one side of the identity.
  - The proof ends with the expression on the other side of the identity.
  - The proof in between consists of showing in sequence a series of expressions, each one easily seen to be equivalent to its preceding expression.
  - Begin with the more complicated expression and work towards the less complicated expression.
  - If no other move presents itself as a good choice, convert the entire expression to one involving only sines and cosines.
  - Combine function by obtaining a common denominator.
  - Factor functions.
  - Use difference of squares $(a+b)(a-b) = a^2 - b^2$ (in either direction) to set up use of Pythagorean identities.
  - Keep in my mind you final destination, and choose steps that bring you closer to the final form you seek.
  - Work from both sides and see if you can meet in the middle.

- The Law of Sines (5.5)

  There are two possibilities for the shape of the triangle created with interior angles A, B, C and sides of length a, b, c. The sides are labelled opposite their corresponding angles. The perpendicular height is labelled h in both cases.



  From either of the diagrams above, we have $\sin A = \dfrac{h}{b}$.

  Also, from the diagram on the left, we have $\sin B = \dfrac{h}{a}$.

  Also, from the diagram on the right, we have $\sin(\pi - B) = \dfrac{h}{a}$.

$$
\begin{aligned}
\sin(u-v) &= \sin u\cos v - \cos u \sin v \\
\sin(\pi - B) &= \sin\pi\cos B - \cos\pi\sin B \\
&= (0)\cos B - (-1)\sin B \\
&= \sin B = \frac{h}{a}
\end{aligned}
$$

Therefore, for both triangles we have

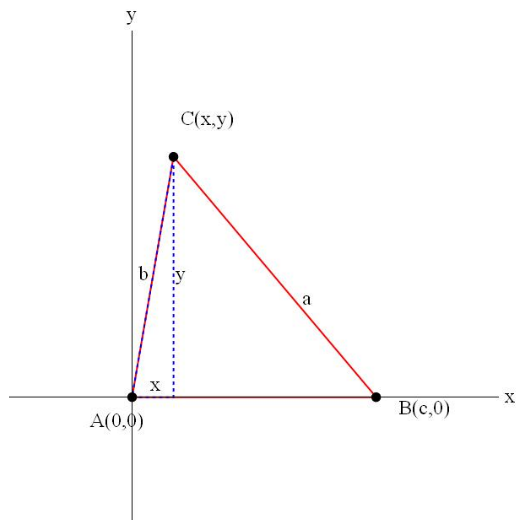
$$\begin{aligned} h = b\sin A &= a\sin B \\ \frac{\sin A}{a} &= \frac{\sin B}{b} \end{aligned}$$

You could do exactly the same thing where you drop the perpendicular to the other two sides.

This leads to the Law of Sines: $\dfrac{\sin A}{a} = \dfrac{\sin B}{b} = \dfrac{\sin C}{c}$

- Law of Cosines (5.6)

  The law of cosines is a generalization of the Pythagorean theorem. It can be derived in a manner similar to how we derived the identity for $\cos(u - v)$.

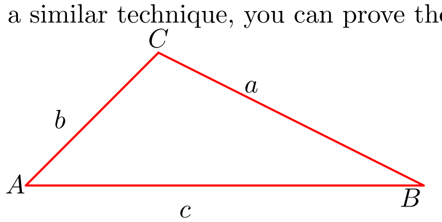


The coordinates of the point C satisfy: $\cos A = \dfrac{x}{b}$ and $\sin A = \dfrac{y}{b}$.

Therefore, $x = b\cos A$ and $y = b\sin A$. Using the distance formula, we can write

$$\begin{aligned} a &= \sqrt{(x-c)^2 + (y-0)^2} \\ a^2 &= (x-c)^2 + y^2 \\ a^2 &= (b\cos A - c)^2 + (b\sin A)^2 \\ a^2 &= b^2\cos^2 A + c^2 - 2bc\cos A + b^2\sin^2 A \\ a^2 &= b^2(\cos^2 A + \sin^2 A) + c^2 - 2bc\cos A \\ a^2 &= b^2(1) + c^2 - 2bc\cos A \\ a^2 &= b^2 + c^2 - 2bc\cos A \end{aligned}$$

Using a similar technique, you can prove the other law of cosines results.



$a^2 = b^2 + c^2 - 2bc\cos A$
$b^2 = a^2 + c^2 - 2ac\cos B$
$c^2 = a^2 + b^2 - 2ab\cos C$