The Language of Mathematics and Common Mathematical Errors

I have created a list of the most common mathematical errors that I have seen occur in calculus. Many of these errors are algebraic or notational rather than calculus errors, yet these errors can prevent you from learning calculus.

I don't want to spend a lecture going over these errors, since many of you will never make a single one of these errors, and most of you may only occasionally make one of these errors. Occasionally making one of these errors is not such a big deal. However, sometimes a student consistently makes these errors, and that *is* a big deal.

You may not understand all the terms used, since some of the concepts refer to things that you will learn as you study calculus.

Read this document, refer back to it if you have to, and most importantly–come talk to me if you have any questions.

These are all serious errors, and you can expect to be penalized if you make them on tests or quizzes.

Improper Use of = Sign

The equal sign should be read as "is equivalent to" in all uses. Some students have developed the habit of thinking the equal sign means "and next I do" or "this implies that", which is incorrect.

Here is an incorrect use of the equal sign:

Incorrect: $$\frac{3x}{2} = 2 = 3x = 4 = x = \frac{4}{3}$$

The problem here is that the equals sign is sometimes being used to mean "which implies" and sometimes to mean "is equivalent to". What the student who wrote the above is probably trying to say is this:

$$\frac{3x}{2} = 2 \underset{\text{which implies that}}{\rightsquigarrow} 3x = 4 \underset{\text{which implies that}}{\rightsquigarrow} x = \frac{4}{3}$$

This could be even better presented if it is written in the following manner, with notes to the side explaining what has been done.

$$\begin{aligned} \frac{3x}{2} &= 2 \\ 3x &= 4 \quad \text{multiply the equation by 2.} \\ x &= \frac{4}{3} \quad \text{solve for } x. \end{aligned}$$

The statements are what the student is doing in their head. Including short phrases that guide the reader (which is usually the person who wrote the solution!) through the process of obtaining the solution is a good idea. Many of the more interesting problems encountered cannot be solved in one line, and the solution may best be presented as groupings of shorter problems which can be combined to give the whole.

It is also incorrect to <u>not</u> include equal signs when working with equivalent expressions.

Incorrect:

$$x^2 + 2x - 1$$
$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$\frac{-2 \pm \sqrt{4 - 4(1)(-1)}}{2}$$
$$-1 \pm \sqrt{2}$$

Correct:

$$\begin{aligned} x^2 + 2x - 1 &= 0 \\ x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-2 \pm \sqrt{4 - 4(1)(-1)}}{2} \\ x &= -1 \pm \sqrt{2} \end{aligned}$$

Improper Algebraic Cancellation

Cancelling a term common to a numerator and denominator can only be done when the term can be factored out of the numerator and denominator. Improper cancellation can transform a problem that is solvable into one which is insolvable!

Incorrect: $\frac{(x+1)(3x+27)+x^3}{x+1} = 3x+27+x^3$

Correct: $\frac{(x+1)(3x+27)+x^3}{x+1} = \frac{\cancel{(x+1)}\left[(3x+27)+\frac{x^3}{x+1}\right]}{\cancel{x+1}} = 3x+27+\frac{x^3}{x+1}$

Improper Use of Brackets

Forgetting brackets can seriously alter a problem. Here are some examples of incorrect mathematical statements, where the bracket is missing:

$$\text{Incorrect: } -\lim_{x\to -1}(x-1) = --1-1 = 1-1 = 0$$

The improper use of brackets frequently shows up, as it does in the problem above, as two algebraic operators written consecutively (i.e., $--$, $-+$, $\times+$, etc.). Anytime you check your solution and see two operators written consecutively, you have made an error! The proper solution retains the brackets as follows:

$$\text{Correct: } -\lim_{x\to -1}(x-1) = -(-1-1) = -(-2) = 2$$

Another example of improper use of brackets frequently occurs in integration. An example is given below.

Incorrect: $\int x-1\,dx.$

Correct: $\int (x-1)\,dx.$

Thinking Everything Commutes

Mathematicians say two operations *commute* if we can perform them in either order and get the same result. Commutivity means we can write things like $6+5=5+6$ and $2\times 4 = 4\times 2$. So, the integers commute under addition and multiplication. However, everything does not commute. Here are some common examples of algebraic errors that result from assuming that everything commutes. For each example, try to identify the two operations have been mistakenly assumed to commute (in the first example, it is the operation of addition and squaring–adding and then squaring is not the same as squaring and then adding).

Incorrect: $(x+y)^2 = x^2+y^2$

Incorrect: $\sin(x+y) = \sin x + \sin y$

Incorrect: $\sqrt{x+y} = \sqrt{x}+\sqrt{y}$

Incorrect: $\frac{1}{x+y} = \frac{1}{x}+\frac{1}{y}$

Incorrect: $\sin 2x = 2\sin x$

Incorrect: $f(x+y) = f(x)+f(y)$

Operators and Improper Ordering

In mathematics there is a natural ordering of things from left to right. For example, the summation sign Σ and the limit are assumed to act on terms to their right.

Here is an example of a statement which suffers from improper ordering.

Incorrect: $\lim_{n\to 0} 2^n(1+n) = 2^n \lim_{n\to 0}(1+n)$.

The equal sign tells us that these two things should be equivalent. However, the limit tells us to take $n \to 0$. Since the factor 2^n depends on n, it cannot be moved to the left of the limit.

Taking a limit and the limit notation is an example of an *operator*. An operator is a object that acts on (or operates on) an expression. Three common operators we will encounter in calculus are limit operators, differential operators and integral operators.

$$\lim_{x\to 2}\left[\frac{x^2}{x^2-4}\right] \text{ (limit operator)}$$
$$\frac{d}{dx}\left[x^2\right] \text{ (differential operator)}$$
$$\int_0^t 2x(x-1)\,dx \text{ (integral operator)}$$

Notice the operator is on the left side of the expression it is operating on.

The integral operator actually has both $\int_0^t$ and dx as part of the operator notation, and we typically write the expression being integrated (operated on) between $\int_0^t$ and dx. The dx tells you that the integration is with respect to x. Therefore, it is possible to move the 2 out of the integral since it does not depend on x,

$$\int 2x(x-1)\,dx = 2\int x(x-1)\,dx.$$

However, you *cannot* remove the factor $2x$, since the x is being integrated over.

Incorrect: $\int 2x(x-1)\,dx = 2x\int (x-1)\,dx$.

We will study this notation in detail in Chapter 5.

It would be incorrect to write:

$$\lim_{x\to 2}\left[\frac{x^2}{x^2-4}\right] \neq \left[\frac{x^2}{x^2-4}\right]\lim_{x\to 2} \text{ (incorrect operator notation)}$$
$$\frac{d}{dx}\left[x^2\right] \neq \left[x^2\right]\frac{d}{dx} \text{ (incorrect operator notation)}$$
$$\int_0^t 2x(x-1)\,dx \neq 2x\int_0^t (x-1)\,dx \text{ (incorrect operator notation)}$$

Sloppy Writing (fractions)

Well, this one is hard to show using typesetting software! Work needs to be written legibly so it can be understood. Also, work needs to be written legibly so errors don't creep into a solution. It is not hard to imagine that without care, it is possible to incorrectly recopy

$$\frac{1}{x+1} \quad \text{as} \quad \frac{1}{x}+1.$$

Conditional Expressions

A *conditional expression* is something that is true only if a condition is satisfied. For example, given the polynomial function $f(x) = -3x^3 + 2x - 1$ we can say that

> If $|x|$ is large, then $f(x) \sim -3x^3$.

What is written above is a conditional expression, since $f(x) \sim -3x^3$ only if the condition that $|x|$ is large is satisfied. The condition is an important part of the statement, since without the condition the conclusion is not true. For example, $f(x)$ not approximately $-3x^3$ if x is close to zero. Note that the concepts of limits will help us make statements like "$|x|$ is large" and "x is near zero" more precise.

You can think of conditional statements as having the following form:

> If (condition), Then (conclusion).

Conditional statements also form the basis of how Theorems can be constructed. When we examine Theorems, pay close attention to the conditions under which the Theorem applies.

Not Checking Your Work

One thing that is not stressed enough is checking your work. Redoing the problem as a check should be done only as a last resort, since if there is a mistake you are liable to make the same mistake twice.

If at all possible, you should check your work by some means that is different than how you solved the problem. And you should check individual steps in a solution as you proceed. Making an error in an algebraic simplification can render a simple integral horribly complicated!

The act of checking also provides practice with mathematics that will help you solve more complicated problems in the future.

- If you do an integral, take the derivative to verify you have the correct formula.
- If you found the roots of an equation, plug the numbers you found back in and make sure they are the roots.
- If you factor a polynomial, multiply the factors together to make sure you factored correctly.
- If you get the inverse of a function, you can check by back substitution that it really is the inverse.

Trying to Memorize Instead of Understanding

Some aspects of calculus need to be memorized, for example derivatives and integrals. But you should work towards understanding the problem, and the general method of constructing a solution. Sometimes I will provide you with something like the "three steps to a solution" (for example, in calculating inverse functions). But most problems in mathematics are complicated, with many parts. It is difficult to see all the steps you will need to take when you begin a problem. However, with practice, you can develop the skills necessary to break large problems into smaller problems, solve the smaller problems, and then put everything back together to solve the larger problem.

The final thought I want to leave you with is this:

> It is not a mistake to write something down incorrectly initially,
> *but it is a mistake to not go back and correct it!*