Questions

Example Find $f(2+h)$, $f(x+h)$, and $\dfrac{f(x+h)-f(x)}{h}$ where $h \neq 0$ and $f(x) = x - x^2$.

Example Find the domain and sketch the graph of the function $g(x) = \sqrt{6-2x}$.

Example Find the domain and sketch the graph of the function $f(x) = \begin{cases} 2x+3 & \text{if } x < -1 \\ 3-x & \text{if } x \geq -1 \end{cases}$.

Example A taxi company charges two dollars for the first mile (or part of a mile) and 20 cents for each succeeding tenth of a mile (or part). Express the cost C (in dollars) of a ride as a function of the distance traveled (in miles) for $0 < x < 2$, and sketch the graph of this function.

Example In a certain country, income tax is assessed as follows. There is no tax on income up to $10,000. Any income over $10,000 is taxed at a rate of 10%, up to an income of $20,000. Any income over $20,000 is taxed at 15%.

a) Sketch the graph of the tax rate R as a function of the income I.

b) How much tax is assessed on an income of $14,000? On $26,000?

c) Sketch the graph of the total assessed tax T as a function of the income I.

Solutions

Example Find $f(2+h)$, $f(x+h)$, and $\dfrac{f(x+h)-f(x)}{h}$ where $h \neq 0$ and $f(x) = x - x^2$.

In this problem we just have to be careful with the algebra. The colour is meant to help you see how the functional substitutions work.

$$f(x) = x - x^2$$

$$\begin{aligned} f(2+h) &= (2+h) - (2+h)^2 \\ &= (2+h) - (4+h^2+4h) \\ &= -2 - h^2 - 3h \end{aligned}$$

$$\begin{aligned} f(x+h) &= (x+h) - (x+h)^2 \\ &= (x+h) - (x^2+h^2+2xh) \\ &= -x^2 - h^2 - 2xh + x + h \end{aligned}$$

$$\begin{aligned} \frac{f(x+h)-f(x)}{h} &= \frac{1}{h}\left(f(x+h) - f(x)\right) \\ &= \frac{1}{h}\left(-x^2 - h^2 - 2xh + x + h - \left(x - x^2\right)\right) \\ &= \frac{1}{h}\left(-x^2 - h^2 - 2xh + x + h - x + x^2\right) \\ &= \frac{1}{h}\left(-h^2 - 2xh + h\right) \end{aligned}$$

$$\begin{aligned} &= \frac{h}{h}\left(-h-2x+1\right) \\ &= \left(-h-2x+1\right), \text{ since } h \neq 0,\ h/h=1. \end{aligned}$$

Example Find the domain and sketch the graph of the function $g(x)=\sqrt{6-2x}$.

First, let's get the domain of $g(x)=\sqrt{6-2x}$. Once we have the domain, we can construct the graph.

We need to use the fact that the square root function $f(y)=\sqrt{y}$ is defined on the real numbers $\mathbb{R}$, only if $y \geq 0$. For the function $g(x)$, this means $6-2x \geq 0$.

$$\begin{aligned} 6-2x &\geq 0 \\ -2x &\geq -6 \\ x &\leq 3 \text{ (dividing by number less than zero changes the inequality)} \end{aligned}$$

So the domain of $g(x)=\sqrt{6-2x}$ is $x \leq 3$. Other ways of writing the domain are $-\infty \leq x \leq 3$, or $x \in (-\infty, 3]$. The range of the function is $g(x) \in [0, \infty)$.

For the sketch, let's first think of what the sketch of the square root function $\sqrt{x}$ looks like. Then I sketched the graph of $\sqrt{-x}$. From these graphs we can construct the sketch of $g(x)$.

You can check that you graph is correct by making sure it crosses the x and y axis at the proper points.

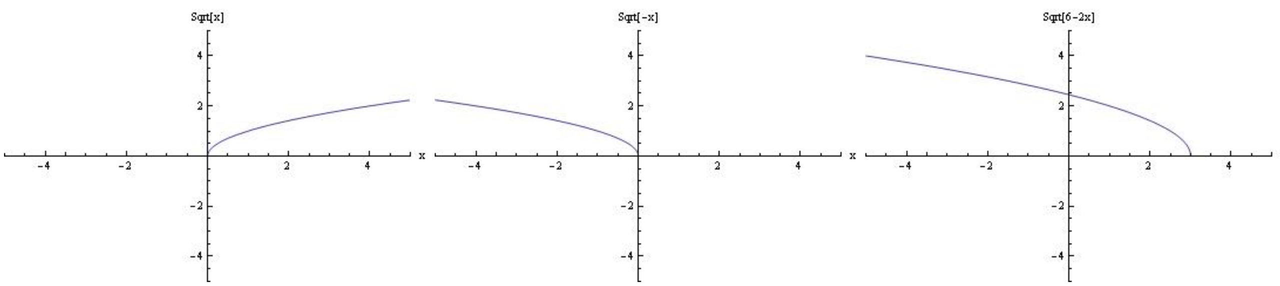

Figure 1: The steps I used to construct the graph of $g(x)=\sqrt{6-2x}$.

Example Find the domain and sketch the graph of the function $f(x)=\begin{cases} 2x+3 & \text{if } x<-1 \\ 3-x & \text{if } x \geq -1 \end{cases}$.

The function $f(x)$ is piecewise defined. Therefore, I am going to plot all the functions we need individually first, and then combine them to get the desired sketch of the graph of $f(x)$.

First, let's graph $y=2x+3$. This is a straight line, with the form $y=mx+b$, where m is the slope and b is the y-intercept. This function has a slope of 2, and a y-intercept of 3.

To sketch the graph of a straight line, all we need are two points on the line. The easiest two points to find are usually the x-intercept and the y-intercept. If $x=0$, then $y=3$, so the point $(0,3)$ is on the line (we knew this above since we had worked out the y-intercept!). If $y=0$, then $x=-3/2$, giving us a second point on the line as $(-3/2,0)$. I've sketched the graph below.

We can do the same thing to get a sketch of the graph of $y=3-x$. Here we identify the two points $(0,3)$ and $(3,0)$ as being on the line.

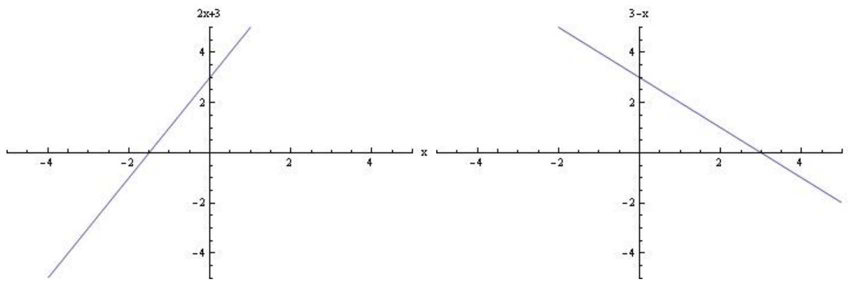

Figure 2: The graph of $y = 2x + 3$ and $y = 3 - x$.

Now, we combine the above two plots to get the final sketch. It is important to identify where the function changes definition, and label as many points as necessary to make your sketch informative. Note that I can't easily fill in holes or show empty holes in my sketches which were created on a computer. The best sketch you could present would look something like the one on the left. Notice that I have labeled some points of interest.

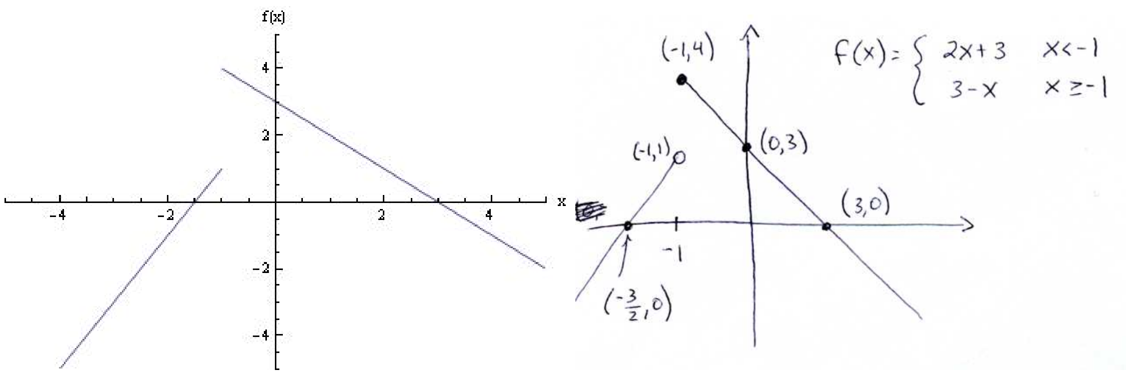

Figure 3: The graph of $f(x)$.

In this case we can get the domain and range from the sketch. This is different from Problem 1, where we got the domain and range first! Here, the domain is $x \in \mathbb{R}$, and the range is $f(x) \leq 4$.

Example A taxi company charges two dollars for the first mile (or part of a mile) and 20 cents for each succeeding tenth of a mile (or part). Express the cost C (in dollars) of a ride as a function of the distance traveled (in miles) for $0 < x < 2$, and sketch the graph of this function.

In this problem we are told what the function is like in words, and need to construct the sketch and analytic form of the function. It is probably easiest to convert the words to a sketch, and then the sketch to an analytic function. The analytic function that relates to this graph is obviously piecewise defined. You could write a long piecewise definition which would

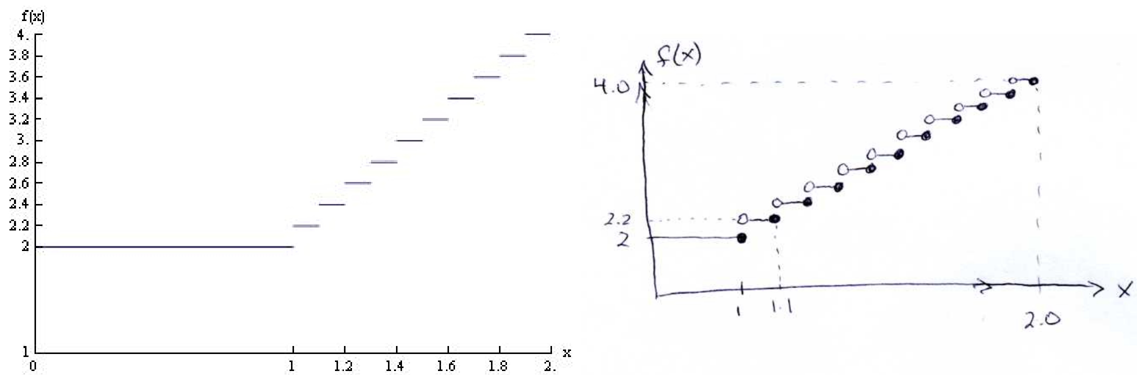

Figure 4: The graph of $f(x)$ for the taxi problem. The computer generated graph is missing the open and filled dots. Your graph should look more like the one on the left.

capture the behaviour of the function in the region $0 < x < 2$:

$$f(x) = \begin{cases} 2 & \text{if } x \in (0,1] \\ 2.20 & \text{if } x \in (1,1.1] \\ 2.40 & \text{if } x \in (1.1,1.2] \\ 2.60 & \text{if } x \in (1.2,1.3] \\ 2.80 & \text{if } x \in (1.3,1.4] \\ 3.00 & \text{if } x \in (1.4,1.5] \\ 3.20 & \text{if } x \in (1.5,1.6] \\ 3.40 & \text{if } x \in (1.6,1.7] \\ 3.60 & \text{if } x \in (1.7,1.8] \\ 3.80 & \text{if } x \in (1.8,1.9] \\ 4.00 & \text{if } x \in (1.9,2.0] \end{cases} .$$

This is very cumbersome. Another way or representing the function would be by using the *greater integer function*, which we will bump into later.

Example In a certain country, income tax is assessed as follows. There is no tax on income up to \$10,000. Any income over \$10,000 is taxed at a rate of 10%, up to an income of \$20,000. Any income over \$20,000 is taxed at 15%.

a) Sketch the graph of the tax rate R as a function of the income I.

b) How much tax is assessed on an income of \$14,000? On \$26,000?

c) Sketch the graph of the total assessed tax T as a function of the income I.

The graph is easily sketched from the information we are given. And from the graph we can obtain the analytic expression for the rate of income tax:

$$\text{Rate of Income Tax } = R(I) = \begin{cases} 0 & \text{if } I \in (0,10000] \\ 10 & \text{if } I \in (10000,20000] \\ 15 & \text{if } I \in (20000,\infty) \end{cases} .$$

To find the tax on an income of \$14,000, we do the following:

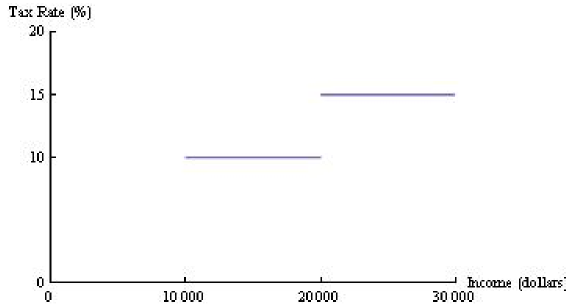

Figure 5: The graph of tax rate $R(I)$ for the income tax problem. Notice that this computer generated graph isn't as good as a hand drawn graph because it is missing the open circles and closed dots that a hand drawn graph would have. There should be circles at $(10000, 10)$ and $(20000, 15)$; there should be filled dots at $(10000, 0)$ and $(20000, 10)$. If you look in you text you will see the more correct sketch.

0% tax on the first \$10,000 earned =	\$0 in tax
10% tax on the remaining \$4,000 earned =	\$400 in tax
	\$400 taxes paid

To find the tax on an income of \$26,000, we do the following:

0% tax on the first \$10,000 earned =	\$0 in tax
10% tax on the next \$10,000 earned =	\$1000 in tax
15% tax on the remaining \$6,000 earned =	\$900 in tax
	\$1900 taxes paid

To get a sketch of the assessed tax as a function of income, we need to think a little bit.

For income below \$10,000 the tax is assessed at a rate of 0% (no tax is collected). This means that the assessed tax function $T(I)$ will be linear in this region, and have a slope of 0.

For income from \$10,000 to \$20,000 the tax is assessed at a rate of 10%. This means that the assessed tax function $T(I)$ will be linear in this region, and have a slope of 0.1.

For income above \$20,000 the tax is assessed at a rate of 15%. This means that the assessed tax function $T(I)$ will be linear in this region, and have a slope of 0.15.

This is combined into the following graph:

Getting the analytic function that this graph represents involves a small amount of work. I present it here for those of you who want to try and get it. Feel free to stop by if you want some hints!

$$\text{Tax Assessed} = T(I) = \begin{cases} 0 & \text{if } I \in (0, 10000] \\ 0.1I - 1000 & \text{if } I \in (10000, 20000] \\ 0.15I - 2000 & \text{if } I \in (20000, \infty) \end{cases} .$$

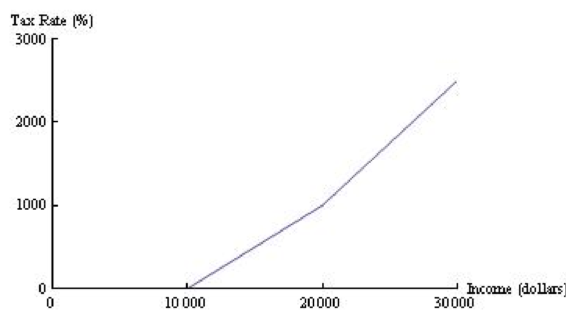

Figure 6: The graph of the assessed tax $T(I)$ for the income tax problem.