Questions

Example Starting with the graph of $y = e^x$, write the equation of the graph that results from

a) shifting 2 units downward,
b) shifting 2 units to the right,
c) reflecting about the x-axis,
d) reflecting about the y-axis,
e) reflecting about the x-axis and then about the y-axis.

Example Starting with the graph of $y = e^x$, find the equation of the graph that results from

a) reflecting about the line $y = 4$,
b) reflecting about the line $x = 2$.

Example Compare the functions $f(x) = x^5$ and $g(x) = 5^x$ by graphing both functions in several viewing rectangles. Find all points of intersection of the graphs correct to one decimal place. Which function grows more rapidly when x is large?

Example An isotope of sodium, ^{2}4Na, has a half-life of 15 hours. A sample of this isotope has mass 2g.

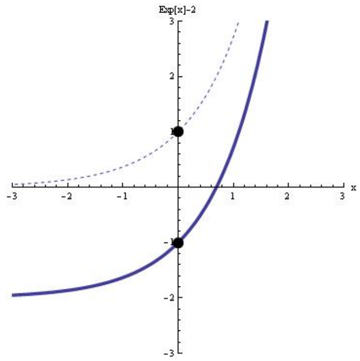
a) Find the amount remaining after 60 hours.
b) Find the amount remaining after t hours.
c) Estimate the amount remaining after 4 days.
d) Use a graph to estimate the time required for the mass to be reduced to 0.01g.

Solutions

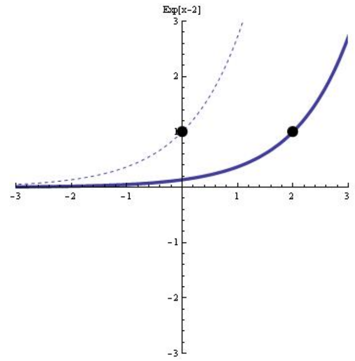
Example Starting with the graph of $y = e^x$, write the equation of the graph that results from

a) shifting 2 units downward,
b) shifting 2 units to the right,
c) reflecting about the x-axis,
d) reflecting about the y-axis,
e) reflecting about the x-axis and then about the y-axis.

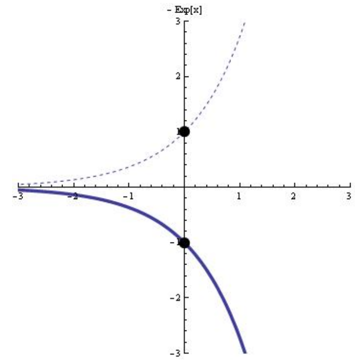
I've included the sketches to help illustrate what is going on. Also, on the sketches I have included the original function $y = e^x$ as a dashed line. The big dot is also included to help orient how the new graph is related to the old.



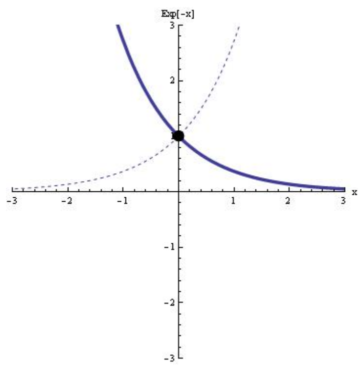
a) $y = e^x - 2$



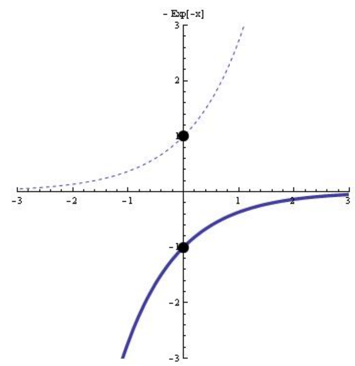
b) $y = e^{x-2}$



c) $y = -e^x$



d) $y = e^{-x}$



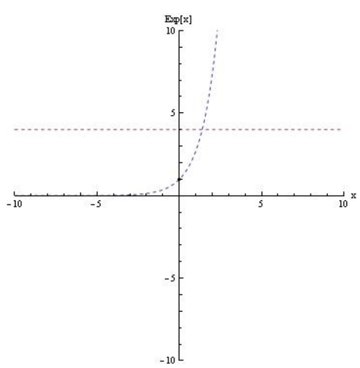
e) $y = -e^{-x}$

Example Starting with the graph of $y = e^x$, find the equation of the graph that results from

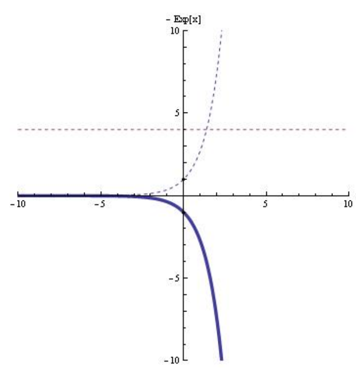
a) reflecting about the line $y = 4$,
b) reflecting about the line $x = 2$.

Again, I am going to include some graphs to show what is happening with the transformations.

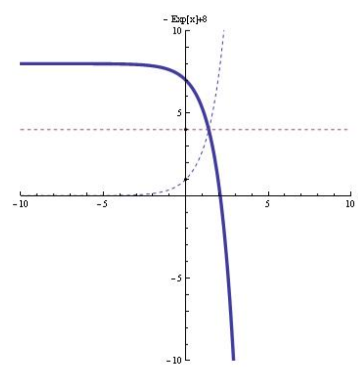
a) We want to end up with the graph $y = e^x$ reflected about the line $y = 4$. We do this in a couple of steps, outlined below.



Begin with the graph $y = e^x$.



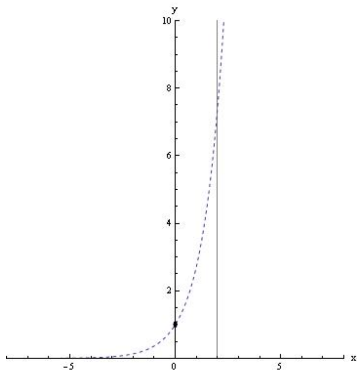
Then, rotate about the x-axis to get $y = -e^x$.



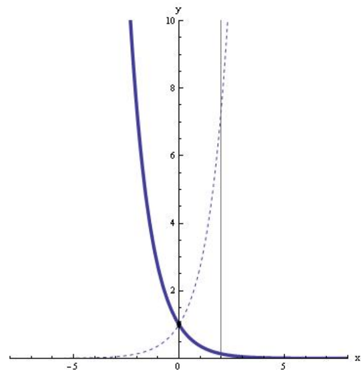
Finally, shift upwards by 8 $y = -e^x + 8$.

The key idea is that the graph should look something like it does in the final graph *before I begin making any transformations*. Once I know the final look, you can search for one, two, three, (*or more!*) transformations to get from the original to the final result.

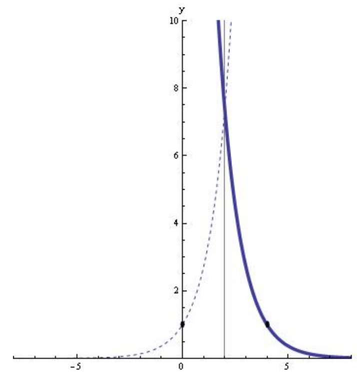
a) We want to end up with the graph $y = e^x$ reflected about the line $x = 2$. We do this in a couple of steps, outlined below.



Begin with the graph $y = e^x$.



Then, rotate about the y-axis to get $y = e^{-x}$.



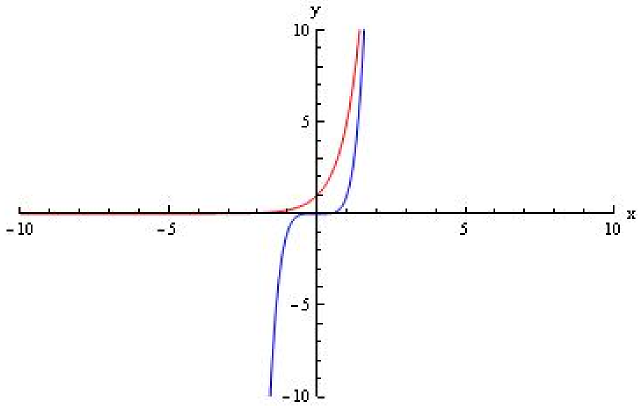
Finally, shift right by 4 $y = e^{-(x-4)} = e^{-x+4}$.

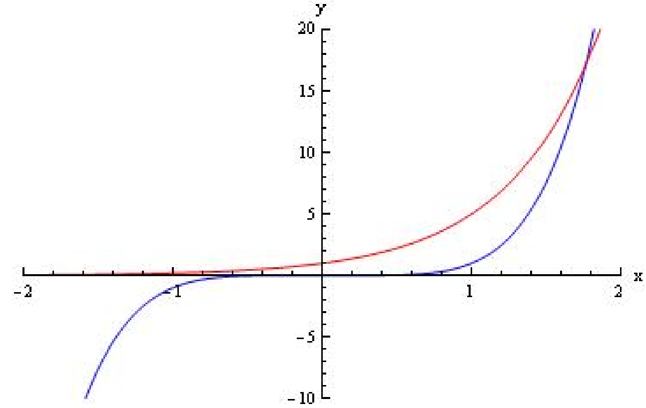
Example Compare the functions $f(x) = x^5$ and $g(x) = 5^x$ by graphing both functions in several viewing rectangles. Find all points of intersection of the graphs correct to one decimal place. Which function grows more rapidly when x is large?

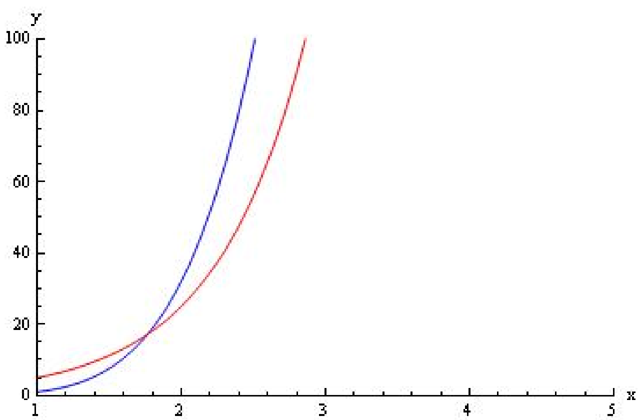
Here we need to use some *Mathematica* commands to generate plots. Here is the command I used for my second plot. Your viewing rectangles may be different than mine.

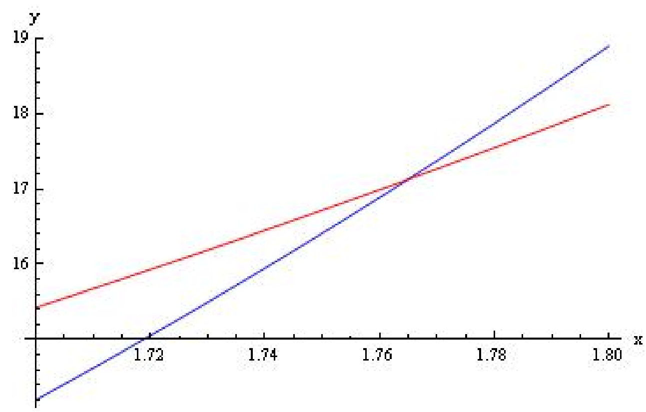
```
Plot[{x^5, 5^x}, {x, -10, 10},
  PlotRange -> {{-2, 2}, {-10, 20}},
  AxesLabel -> {"x", "y"},
  PlotStyle -> {{RGBColor[0, 0, 1]}, {RGBColor[1, 0, 0]}}]
```

In my plots, $y = x^5$ is blue, and $y = 5^x$ is red.





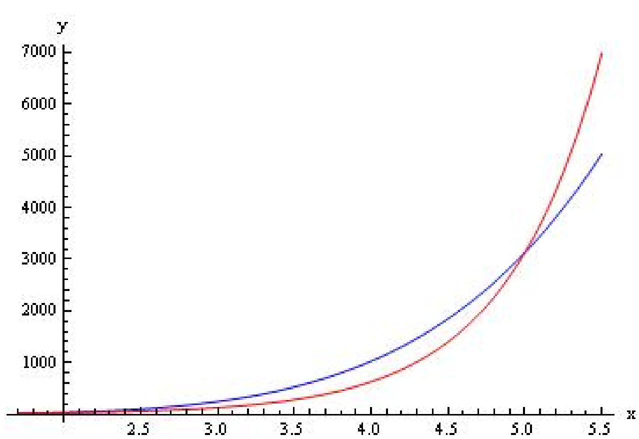


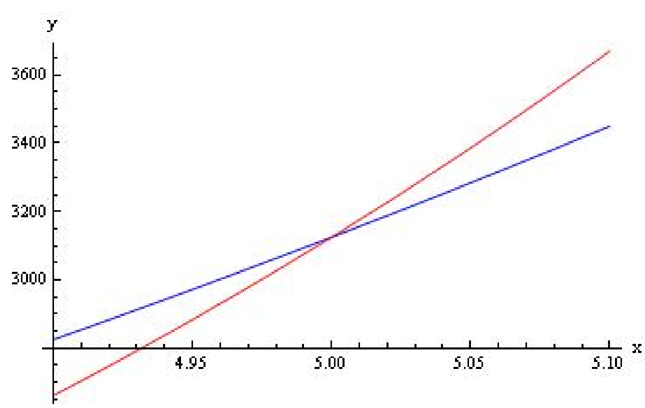


The things I notice about the two functions are that x^5 is odd, increasing, and passes through the origin, while 5^x is neither odd nor even, increasing, and does not pass through the origin. These observations make sense, since x^5 is a polynomial, and 5^x is an exponential function.

The exponential function 5^x grows faster for large x than the polynomial x^5. It appears they intersect once, and by "zooming in" on the graph I estimate that they intersect at approximately $x = 1.8$ (correct to one decimal with rounding). Notice that when you zoom in sufficiently close both graphs appear to be linear.

But do they really intersect only once? I have only examined these functions in a very small area! A little bit more playing, and I discover that they also intersect somewhere else. With a little "zooming", I estimate the second intersection occurs when $x = 5.0$.





The points of intersection occur when the two functions are equal, so $5^x = x^5$. That is hard to solve (!!), but I can see by inspection that $x = 5$ is going to be a point of intersection.

I found two points of intersection graphically, at $x = 1.8$ and $x = 5$.

Example An isotope of sodium, ^{24}Na, has a half-life of 15 hours. A sample of this isotope has mass 2g.

a) Find the amount remaining after 60 hours.
b) Find the amount remaining after t hours.
c) Estimate the amount remaining after 4 days.
d) Use a graph to estimate the time required for the mass to be reduced to 0.01g.

Using the concept of half-life, let's write down the mass remaining at certain time intervals and then try to recognize any patterns. When we are trying to recognize patterns we do not simplify our results.

Let $m(t)$ be the amount of sodium (in g) left at time t (in hours). We are told the half-life is 15 hours, so after each 15 hour period there will be half the mass remaining. That is,

$$\begin{aligned} m(0) &= 2 \\ m(15) &= \frac{1}{2} \cdot 2 \\ m(30) &= \frac{1}{2} \cdot \frac{1}{2} \cdot 2 = \frac{1}{2^2} \cdot 2 \\ m(45) &= \frac{1}{2^3} \cdot 2 \\ m(60) &= \frac{1}{2^4} \cdot 2 \end{aligned}$$

Now we can answer the first question. After 60 hours the amount of sodium left is $2/2^4 = 1/8 = 0.125$ g. To find the amount remaining after t hours, we need to look for the pattern. I am going to rewrite the above so it is easier to recognize the pattern.

$$\begin{aligned} m(0) &= \frac{1}{2^0} \cdot 2 \qquad t = 0 = 0 \cdot 15, \text{ exponent} = 0 \\ m(15) &= \frac{1}{2^1} \cdot 2 \qquad t = 15 = 1 \cdot 15, \text{ exponent} = 1 \\ m(30) &= \frac{1}{2^2} \cdot 2 \qquad t = 30 = 2 \cdot 15, \text{ exponent} = 2 \\ m(45) &= \frac{1}{2^3} \cdot 2 \qquad t = 45 = 3 \cdot 15, \text{ exponent} = 3 \\ m(60) &= \frac{1}{2^4} \cdot 2 \qquad t = 60 = 4 \cdot 15, \text{ exponent} = 4 \end{aligned}$$

From this, it is apparent that the general equation for the amount remaining after t hours must be

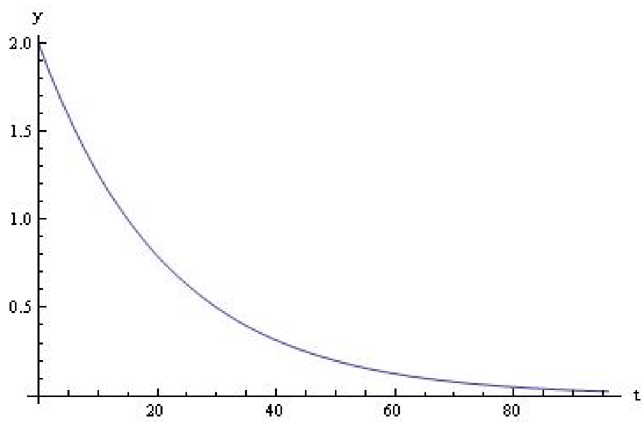
$$m(t) = 2 \cdot \frac{1}{2^{t/15}} = 2 \cdot 2^{-t/15}.$$

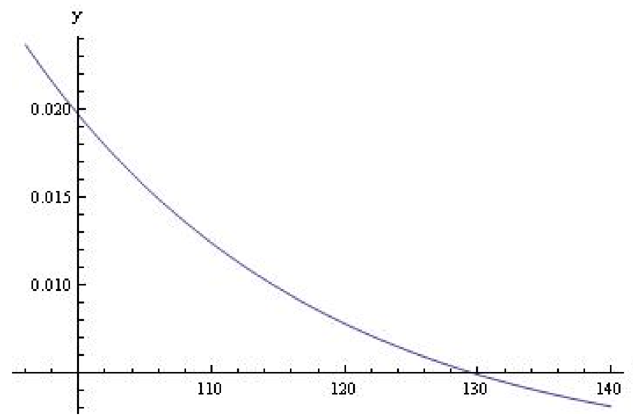
You can substitute in $t = 0, 15, 30, 45, 60$ to verify that this is correct.

After 4 days, which is $4 \cdot 24 = 96$ hours, we have $m(96) = 2 \cdot 2^{-96/15} = 0.0236$ g remaining.

For the graph, you should use *Mathematica*. Here are the commands I used to produce the following graphs (you don't need to do the axis label on the computer, you can just add it in when you copy the graph into your notes):

```
Plot[{2*2^(-t/15)}, {t, 0, 96}, AxesLabel -> {"t", "y"}]
Plot[{2*2^(-t/15)}, {t, 96, 140}, AxesLabel -> {"t", "y"}]
```





From the second graph, it looks like the mass is reduced to less than 0.01g after about 115 hours.