Questions

Example Find a formula for the inverse of the function $f(x) = \dfrac{4x-1}{2x+3}$.

Example Find a formula for the inverse of the function $f(x) = 5 - 4x^3$.

Example Find a formula for the inverse of the function $y = \dfrac{1+e^x}{1-e^x}$

Example Graph the given functions on a common screen. How are these graphs related?

$$y = \log_{1.5} x, \quad y = \ln x, \quad y = \log_{10} x, \quad y = \log_{50} x$$

Example Starting with the graph of $y = \ln x$, find the equation of the graph that results from

a) shifting 3 units upward,
b) shifting 3 units to the left,
c) reflecting about the x-axis,
d) reflecting about the y-axis,
e) reflecting about the line $y = x$,
f) reflecting about the x-axis and then about the line $y = x$,
g) reflecting about the y-axis and then about the line $y = x$.

Solutions

Example Find a formula for the inverse of the function $f(x) = \dfrac{4x-1}{2x+3}$.

There are three steps to finding the inverse of a function:

Step 1) $f(x) = y = \frac{4x-1}{2x+3}$.

Step 2) Solve for x in terms of y:

$$\begin{aligned} y &= \frac{4x-1}{2x+3} \\ y(2x+3) &= 4x-1 \\ 2xy - 4x &= -3y-1 \\ x(2y-4) &= -3y-1 \\ x &= \frac{-3y-1}{2y-4} \\ x &= \frac{3y+1}{4-2y} \end{aligned}$$

Step 3) Interchange x and y. This gives us $y = f^{-1}(x) = \dfrac{3x+1}{4-2x}$.

Let's verify that we have the correct solution by checking the cancellation equations:

$$\begin{aligned}
f\big(f^{-1}(x)\big) &= f\left(\frac{3x+1}{4-2x}\right) \\
&= \frac{4\left(\frac{3x+1}{4-2x}\right)-1}{2\left(\frac{3x+1}{4-2x}\right)+3} \\
&= \frac{\frac{12x+4}{4-2x}-1}{\frac{6x+2}{4-2x}+3} \\
&= \frac{\frac{12x+4}{4-2x}-\frac{4-2x}{4-2x}}{\frac{6x+2}{4-2x}+\frac{12-6x}{4-2x}} \\
&= \frac{\left(\frac{12x+4-4+2x}{4-2x}\right)}{\left(\frac{6x+2+12-6x}{4-2x}\right)} \\
&= \frac{\left(\frac{14x}{4-2x}\right)}{\left(\frac{14}{4-2x}\right)} \\
&= \frac{14x}{14} \\
f\big(f^{-1}(x)\big) &= x \\
f^{-1}\big(f(x)\big) &= f^{-1}\left(\frac{4x-1}{2x+3}\right) \\
&= \frac{3\frac{4x-1}{2x+3}+1}{4-2\frac{4x-1}{2x+3}} \\
&= \frac{\frac{12x-3}{2x+3}+1}{4-\frac{8x-2}{2x+3}} \\
&= \frac{\frac{12x-3}{2x+3}+\frac{2x+3}{2x+3}}{\frac{8x+12}{2x+3}-\frac{8x-2}{2x+3}} \\
&= \frac{\left(\frac{12x-3+2x+3}{2x+3}\right)}{\left(\frac{8x+12-8x+2}{2x+3}\right)} \\
&= \frac{\left(\frac{14x}{2x+3}\right)}{\left(\frac{14}{2x+3}\right)} \\
&= \frac{14x}{14} \\
f^{-1}\big(f(x)\big) &= x
\end{aligned}$$

So the cancellation conditions are satisfied, and we have found the inverse function correctly.

Example Find a formula for the inverse of the function $f(x) = 5 - 4x^3$.

There are three steps to finding the inverse of a function:

Step 1) $f(x) = y = 5 - 4x^3$.

Step 2) Solve for x in terms of y:

$$\begin{aligned}
y &= 5 - 4x^3 \\
y - 5 &= -4x^3 \\
\frac{5-y}{4} &= x^3 \\
\left(\frac{5-y}{4}\right)^{1/3} &= x \\
x &= \left(\frac{5-y}{4}\right)^{1/3}
\end{aligned}$$

Step 3) Interchange x and y. This gives us $y = f^{-1}(x) = \left(\frac{5-x}{4}\right)^{1/3}$.

Let's verify that we have the correct solution by checking the cancellation equations:

$$\begin{aligned}
f\big(f^{-1}(x)\big) &= f\left(\left(\frac{5-x}{4}\right)^{1/3}\right) \\
&= 5 - 4\left(\left(\frac{5-x}{4}\right)^{1/3}\right)^3 \\
&= 5 - 4\left(\frac{5-x}{4}\right) \\
&= 5 - (5 - x) \\
&= 5 - 5 + x \\
f\big(f^{-1}(x)\big) &= x \\
f^{-1}\big(f(x)\big) &= f^{-1}\left(5 - 4x^3\right) \\
&= \left(\frac{5 - (5 - 4x^3)}{4}\right)^{1/3} \\
&= \left(\frac{5 - 5 + 4x^3}{4}\right)^{1/3} \\
&= \left(\frac{4x^3}{4}\right)^{1/3} \\
&= \left(x^3\right)^{1/3} \\
f^{-1}\big(f(x)\big) &= x
\end{aligned}$$

So the cancellation conditions are satisfied, and we have found the inverse function correctly.

Example Find a formula for the inverse of the function $y = \frac{1+e^x}{1-e^x}$

There are three steps to finding the inverse of a function:

Step 1) $f(x) = y = \frac{1+e^x}{1-e^x}$. This was already done for us.

Step 2) Solve for x in terms of y:

$$\begin{aligned}
y &= \frac{1+e^x}{1-e^x} \\
y(1-e^x) &= 1+e^x \\
y - ye^x &= 1+e^x \\
y-1 &= e^x + ye^x \\
y-1 &= e^x(1+y) \\
e^x &= \frac{y-1}{y+1} \\
x &= \ln\left(\frac{y-1}{y+1}\right)
\end{aligned}$$

Step 3) Interchange x and y. This gives us $y = f^{-1}(x) = \ln\left(\dfrac{x-1}{x+1}\right)$.

Let's verify that we have the correct solution by checking the cancellation equations:

$$\begin{aligned}
f\big(f^{-1}(x)\big) &= f\left(\ln\left(\frac{x-1}{x+1}\right)\right) \\
&= \frac{1+\exp\left(\ln\left(\frac{x-1}{x+1}\right)\right)}{1-\exp\left(\ln\left(\frac{x-1}{x+1}\right)\right)} \\
&= \frac{1+\left(\frac{x-1}{x+1}\right)}{1-\left(\frac{x-1}{x+1}\right)} \\
&= \frac{x+1+x-1}{x+1-x+1} \\
&= \frac{2x}{2} \\
f\big(f^{-1}(x)\big) &= x \\
f^{-1}\big(f(x)\big) &= f^{-1}\left(\frac{1+e^x}{1-e^x}\right) \\
&= \ln\left(\frac{\frac{1+e^x}{1-e^x}-1}{\frac{1+e^x}{1-e^x}+1}\right) \\
&= \ln\left(\frac{1+e^x-(1-e^x)}{1+e^x+(1-e^x)}\right) \\
&= \ln\left(\frac{1+e^x-1+e^x}{1+e^x+1-e^x)}\right) \\
&= \ln\left(\frac{2e^x}{2}\right) \\
&= \ln\left(e^x\right) \\
f^{-1}\big(f(x)\big) &= x
\end{aligned}$$

So the cancellation conditions are satisfied, and we have found the inverse function correctly.

Example Graph the given functions on a common screen. How are these graphs related?

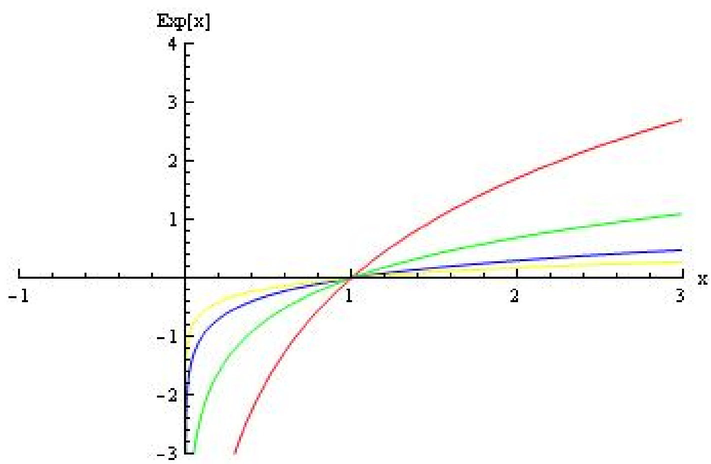
$$y = \log_{1.5} x, \quad y = \ln x, \quad y = \log_{10} x, \quad y = \log_{50} x$$

The functions can be graphed using the following *Mathematica* commands:

```
Plot[{Log[1.5, x], Log[x], Log[10, x], Log[50, x]}, {x, -5, 5},
  PlotRange -> {{-1, 3}, {-3, 4}}]
```

I added options to get the plots to be different colours, and to have the axes labeled. The commands I used to generate the plot below was:

```
Plot[{Log[1.5, x], Log[x], Log[10, x], Log[50, x]}, {x, -5, 5},
  PlotRange -> {{-1, 3}, {-3, 4}}, AxesLabel -> {"x", "Exp[x]"},
  PlotStyle -> {{RGBColor[1, 0, 0]}, {RGBColor[0, 1, 0]},
               {RGBColor[0, 0, 1]}, {RGBColor[1, 1, 0]}}]
```



In my plots, the functions are:

$y = \log_{1.5} x$	red
$y = \ln x$	green
$y = \log_{10} x$	blue
$y = \log_{50} x$	yellow

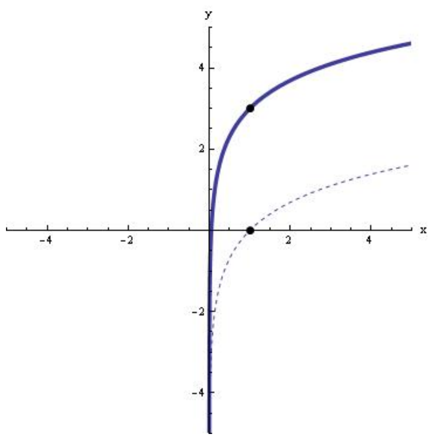
All the plots pass through the point $(1, 0)$, all increase, and all approach negative infinity as x approaches zero from the left. As the base increases, the function stays closer to zero.

Example Starting with the graph of $y = \ln x$, find the equation of the graph that results from

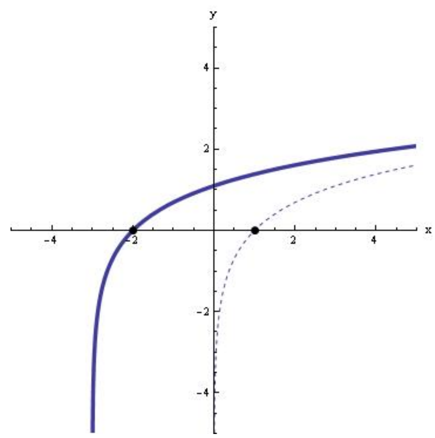
a) shifting 3 units upward,
b) shifting 3 units to the left,
c) reflecting about the x-axis,
d) reflecting about the y-axis,
e) reflecting about the line $y = x$,

f) reflecting about the x-axis and then about the line $y = x$,
g) reflecting about the y-axis and then about the line $y = x$.

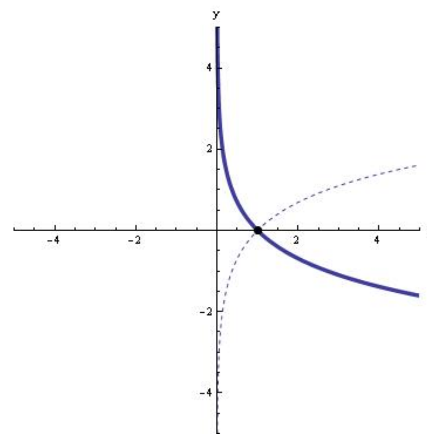
I have plotted the graphs with the reflections below.



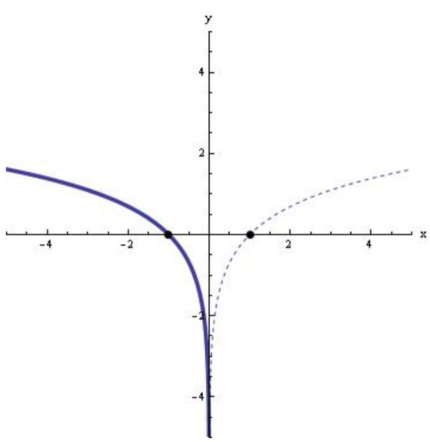
a) $y = \ln x + 3$



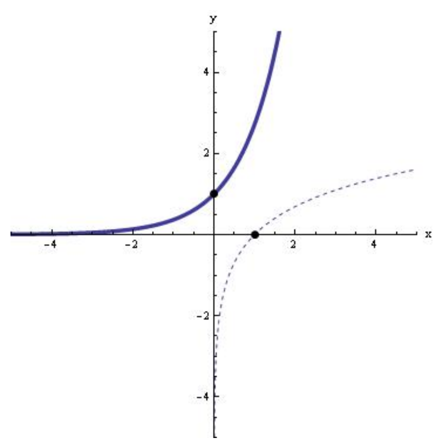
b) $y = \ln(x + 3)$



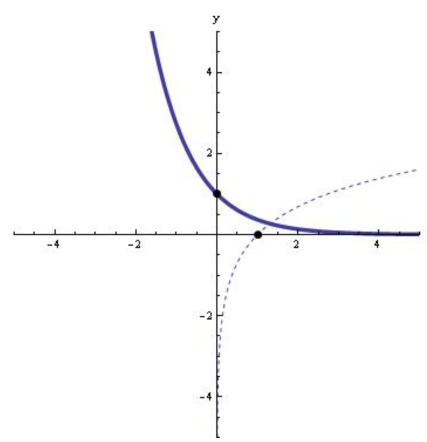
c) $y = -\ln x$



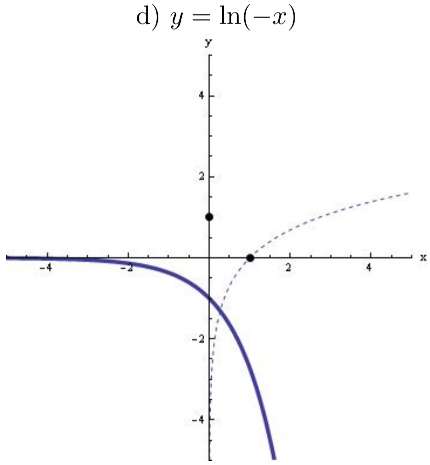
d) $y = \ln(-x)$



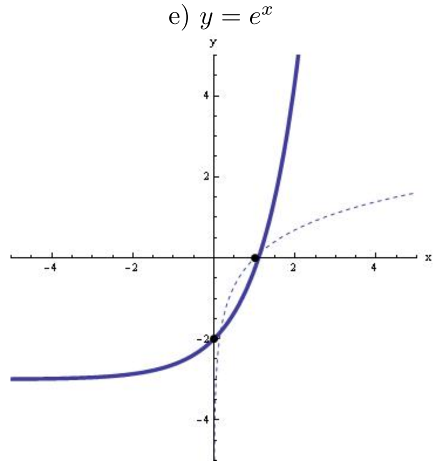
e) $y = e^x$



f) $y = e^{-x}$



g) $y = -e^x$



h) $y = e^x - 3$