Questions

Example The point $P(1, 1/2)$ lies on the curve $y = x/(1+x)$.

a) If Q is the point $(x, x/(1+x))$, use *Mathematica* to find the slope of the secant line PQ correct to six decimal places for the following values of x:
i) 0.5 ii) 0.9 iii) 0.99 iv) 0.999 v) 1.5 vi) 1.1 vii) 1.01 viii) 1.001
b) Using the results of part a), guess the value of the slope of the tangent line to the curve at $P(1, 1/2)$.
c) Using the slope from part b), find an equation of the tangent line to the curve at $P(1, 1/2)$.

Example The point $P(4, 2)$ lies on the curve $y = \sqrt{x}$.

a) If Q is the point $(x, \sqrt{x})$, use *Mathematica* to find the slope of the secant line PQ correct to six decimal places for the following values of x:
i) 5 ii) 4.5 iii) 4.1 iv) 4.01 v) 4.001 vi) 3 vii) 3.5 viii) 3.9 ix) 3.99 x) 3.999
b) Using the results of part a), guess the value of the slope of the tangent line to the curve at $P(4, 2)$.
c) Using the slope from part b), find an equation of the tangent line to the curve at $P(4, 2)$.

Example If a ball is thrown into the air with a velocity of 40 ft/s, its height in feet after t seconds is given by $y = 40t - 16t^2$.

a) Find the average velocity for the time period beginning when $t = 2$ and lasting
i) 0.5 s ii) 0.1 s iii) 0.05 s iv) 0.01 s
b) Find the instantaneous velocity when $t = 2$.

Example The displacement in feet of a certain particle moving in a straight line is given by $s = t^3/6$, where t is measured in seconds.

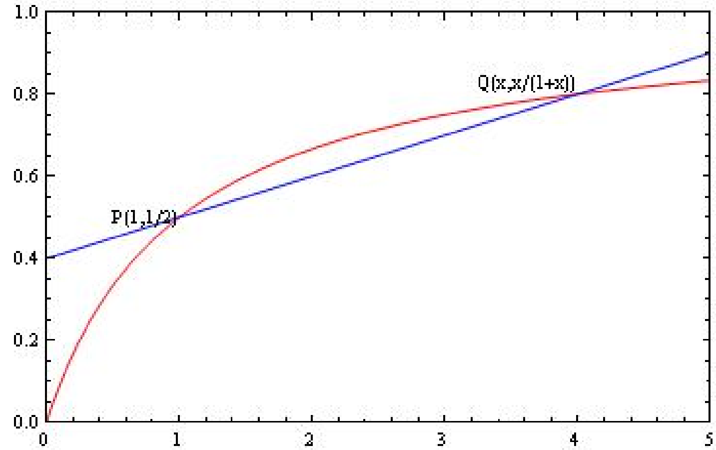
a) Find the average velocity over the following time periods:
i) $[1, 3]$ ii) $[1, 2]$ iii) $[1, 1.15]$ iv) $[1, 1.1]$
b) Find the instantaneous velocity when $t = 1$.
c) Draw the graph of s as a function of t and draw the secant lines whose slopes are the average velocities found in part a).
d) Draw the tangent line whose slope is the instantaneous velocity from part b).

Solutions

Example The point $P(1, 1/2)$ lies on the curve $y = x/(1+x)$.

a) If Q is the point $(x, x/(1+x))$, use *Mathematica* to find the slope of the secant line PQ correct to six decimal places for the following values of x:
i) 0.5 ii) 0.9 iii) 0.99 iv) 0.999 v) 1.5 vi) 1.1 vii) 1.01 viii) 1.001
b) Using the results of part a), guess the value of the slope of the tangent line to the curve at $P(1, 1/2)$.
c) Using the slope from part b), find an equation of the tangent line to the curve at $P(1, 1/2)$.

OK, here we are looking at secant lines and tangents to the function. I've included a sketch below.



The points are $P(1, 1/2)$ and $Q(x, x/(1+x))$. Then the slope of the line passing between PQ is:

$$\begin{aligned} m_{PQ} &= \frac{y_Q - y_P}{x_Q - x_P} \\ &= \frac{x/(1+x) - 1/2}{x-1} \\ &= \frac{\left(\frac{2x-x-1}{2(1+x)}\right)}{x-1} \\ &= \frac{1}{2}\left(\frac{x-1}{1+x}\right)\frac{1}{x-1} \\ &= \frac{1}{2(1+x)} \end{aligned}$$

We can substitute in the values for x to determine the slopes of the secant lines. There are many ways to do this in *Mathematica*. I chose to do it the following way:

```
mPQ[x_] = 1/2/(1 + x)
mPQ[0.5]
mPQ[0.9]
```

x in $Q(x, x/(1+x))$	m_{PQ}
0.5	0.3333
0.9	0.263158
0.99	0.251256
0.999	0.250125

x in $Q(x, x/(1+x))$	m_{PQ}
1.5	0.2
1.1	0.238095
1.01	0.248756
1.001	0.249875

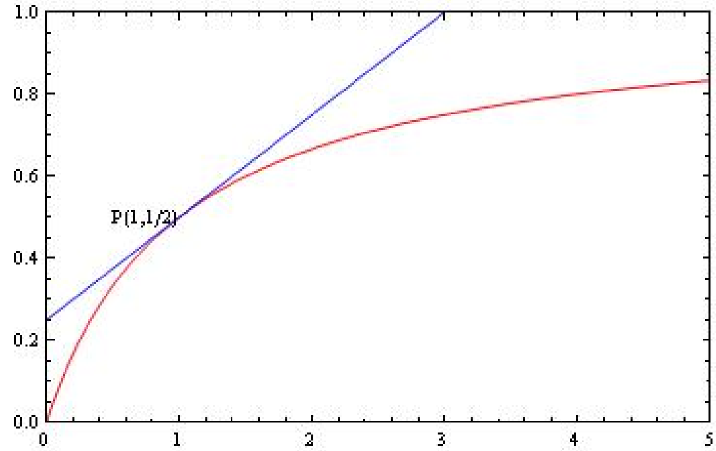
From the tables, I would guess the value of the slope of the tangent line to the curve at $P(1, 1/2)$ would be 0.25.

To get the equation of the tangent line, we can use the point-slope equation for a line. We have guessed that the slope is 0.25, and a point on the line will be $(x_0, y_0) = (1, 1/2)$. Therefore, $m = 0.25 = 1/4$, $x_0 = 1$, $y_0 = 1/2$, and we have

$$y - y_0 = m(x - x_0) \quad \text{point-slope equation of a line}$$

$$\begin{aligned} y &= m(x - x_0) + y_0 \\ &= \frac{1}{4}(x-1) + 1/2 \\ &= \frac{x}{4} + 1/4 \end{aligned}$$

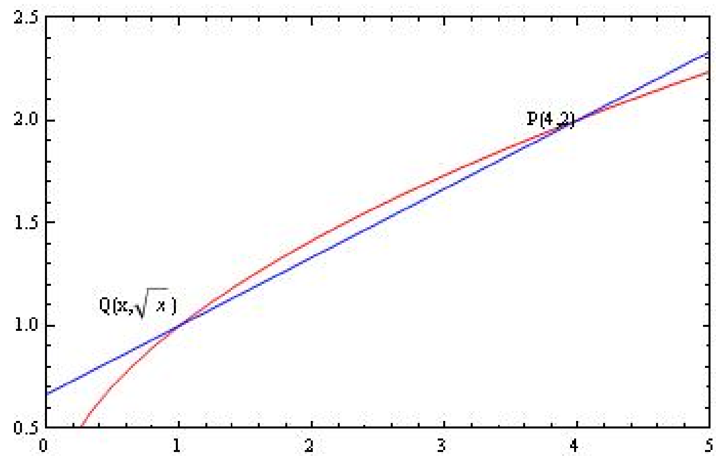
I've included a plot of the function $y = x/(1+x)$ and the tangent line $y = x/4 + 1/4$ below as a check to make sure I have the right answer. Everything looks good!



Example The point $P(4,2)$ lies on the curve $y = \sqrt{x}$.

a) If Q is the point $(x, \sqrt{x})$, use *Mathematica* to find the slope of the secant line PQ correct to six decimal places for the following values of x:
i) 5 ii) 4.5 iii) 4.1 iv) 4.01 v) 4.001 vi) 3 vii) 3.5 viii) 3.9 ix) 3.99 x) 3.999
b) Using the results of part a), guess the value of the slope of the tangent line to the curve at $P(4,2)$.
c) Using the slope from part b), find an equation of the tangent line to the curve at $P(4,2)$.

OK, here we are looking at secant lines and tangents to the square root function. I've included a sketch below.



The points are $P(4,2)$ and $Q(x,\sqrt{x})$. Then the slope of the line passing between PQ is:

$$m_{PQ} = \frac{y_Q - y_P}{x_Q - x_P} = \frac{2-\sqrt{x}}{4-x}$$

We can substitute in the values for x to determine the slopes of the secant lines. There are many ways to do this in *Mathematica*. I chose to do it the following way:

```
N[(2 - Sqrt[x])/(4 - x)] /. x -> {5, 4.5, 4.1, 4.01, 4.001}
```

x in $Q(x,\sqrt{x})$	m_{PQ}
5	0.2361
4.5	0.2426
4.1	0.2485
4.01	0.2498
4.001	0.24998

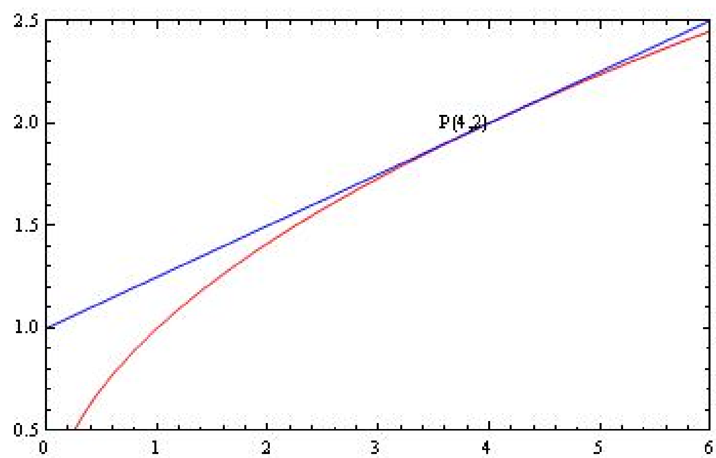
x in $Q(x,\sqrt{x})$	m_{PQ}
3	0.2679
3.5	0.2583
3.9	0.2516
3.99	0.2502
3.999	0.25002

From the tables, I would guess the value of the slope of the tangent line to the curve at $P(4,2)$ would be 0.25.

To get the equation of the tangent line, we can use the point-slope equation for a line. We have guessed that the slope is 0.25, and a point on the line will be $(x_0, y_0) = (4,2)$. Therefore, $m = 0.25 = 1/4$, $x_0 = 4$, $y_0 = 2$, and we have

$$\begin{aligned} y - y_0 &= m(x - x_0) \quad \text{point-slope equation of a line} \\ y &= m(x-x_0) + y_0 \\ &= \frac{1}{4}(x-4) + 2 \\ &= \frac{x}{4} + 1 \end{aligned}$$

I've included a plot of the function $y = \sqrt{x}$ and the tangent line $y = x/4 + 1$ below as a check to make sure I have the right answer. Everything looks good!



Example If a ball is thrown into the air with a velocity of 40 ft/s, its height in feet after t seconds is given by $y = 40t - 16t^2$.

a) Find the average velocity for the time period beginning when $t = 2$ and lasting
i) 0.5 s ii) 0.1 s iii) 0.05 s iv) 0.01 s
b) Find the instantaneous velocity when $t = 2$.

This question involves the average and instantaneous velocities for a ball which moves according to the relation $s(t) = 40t - 16t^2$. Here, s is the height of the ball (in feet) after t seconds have elapsed. The ball was thrown upward with a certain velocity, and that information has already been incorporated into the equation for $s(t)$.

The average and instantaneous velocities are given by the following:

$$\begin{aligned}
\text{average velocity in interval } [t_1, t_2] &= \frac{\text{distance traveled}}{\text{elapsed time}} \\
&= \frac{s(t_2) - s(t_1)}{t_2 - t_1} \\
&= \frac{40t_2 - 16t_2^2 - 40t_1 + 16t_1^2}{t_2 - t_1} \\
\text{instantaneous velocity in interval } [t_1, t_2] &= \lim_{\text{elapsed time} \to 0} \frac{\text{distance traveled}}{\text{elapsed time}} \\
&= \lim_{t_2 \to t_1} \frac{s(t_2) - s(t_1)}{t_2 - t_1} \\
&= \lim_{t_2 \to t_1} \frac{40t_2 - 16t_2^2 - 40t_1 + 16t_1^2}{t_2 - t_1}
\end{aligned}$$

Notice that for the elapsed time to go to zero, we have the final time (t_2) approach the initial time (t_1) in the interval.

Table of limiting procedure on average velocity

elapsed time interval $[t_1, t_2]$	average velocity
$[2, 2.5]$	-32.0
$[2, 2.1]$	-25.6
$[2, 2.05]$	-24.8
$[2, 2.01]$	-24.16

I got the values for the table above with the help of *Mathematica*. The commands I used were the following:

```
v[t1_, t2_] = (40t2 - 16t2^2 - 40t1 + 16t1^2)/(t2 - t1)
v[2, 2.5]
v[2, 2.1]
v[2, 2.05]
v[2, 2.01]
v[2, 2.000000000001]
v[2, 1.999999999999]
```

Based on the above table, the instantaneous velocity appears to be -24 ft/s at time $t = 2$ s. The negative sign means the ball is falling back towards the earth.

Example The displacement in feet of a certain particle moving in a straight line is given by $s = t^3/6$, where t is measured in seconds.

a) Find the average velocity over the following time periods:
i) $[1, 3]$ ii) $[1, 2]$ iii) $[1, 1.15]$ iv) $[1, 1.1]$
b) Find the instantaneous velocity when $t = 1$.
c) Draw the graph of s as a function of t and draw the secant lines whose slopes are the average velocities found in part a).
d) Draw the tangent line whose slope is the instantaneous velocity from part b).

This question also involves the average and instantaneous velocities, this time for a particle which moves according to the relation $s(t) = t^3/6$. Here, s is the displacement (in feet) after t seconds have elapsed.

The average and instantaneous velocities are given by the following:

$$\begin{aligned}
\text{average velocity in interval } [t_1, t_2] &= \frac{\text{distance traveled}}{\text{elapsed time}} \\
&= \frac{s(t_2) - s(t_1)}{t_2 - t_1} \\
&= \frac{t_2^3/6 - t_1^3/6}{t_2 - t_1} \\
&= \frac{t_2^3 - t_1^3}{6(t_2 - t_1)} \\
\text{instantaneous velocity in interval } [t_1, t_2] &= \lim_{\text{elapsed time}\to 0} \frac{\text{distance traveled}}{\text{elapsed time}} \\
&= \lim_{t_2 \to t_1} \frac{s(t_2) - s(t_1)}{t_2 - t_1} \\
&= \lim_{t_2 \to t_1} \frac{t_2^3 - t_1^3}{6(t_2 - t_1)}
\end{aligned}$$

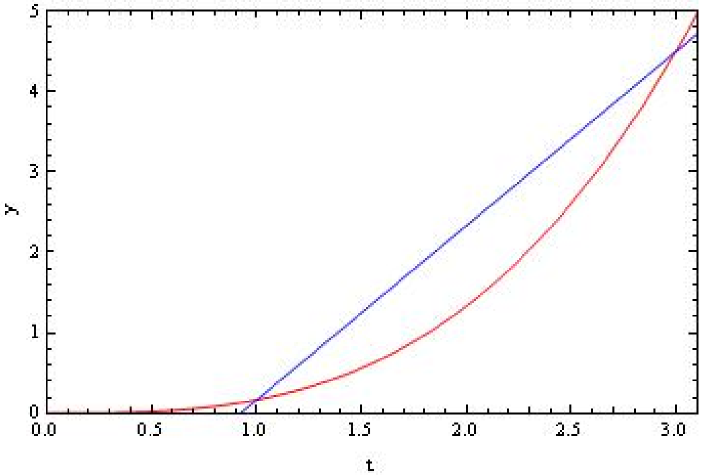
Notice that for the elapsed time to go to zero, we have the final time (t_2) approach the initial time (t_1) in the interval.

Table of limiting procedure on average velocity

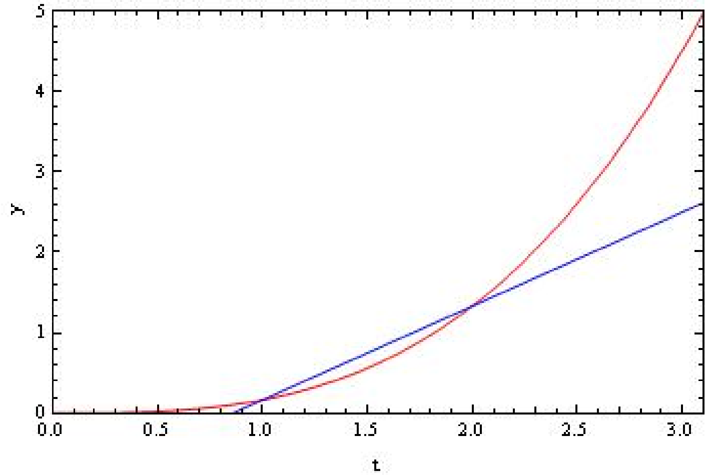
elapsed time interval $[t_1, t_2]$	average velocity
$[1, 3]$	2.16667
$[1, 2]$	1.16667
$[1, 1.5]$	0.791667
$[1, 1.1]$	0.551667
$[1, 1.001]$	0.5005

Based on the above table, the instantaneous velocity appears to be 0.5 ft/s at time $t = 1$ s.

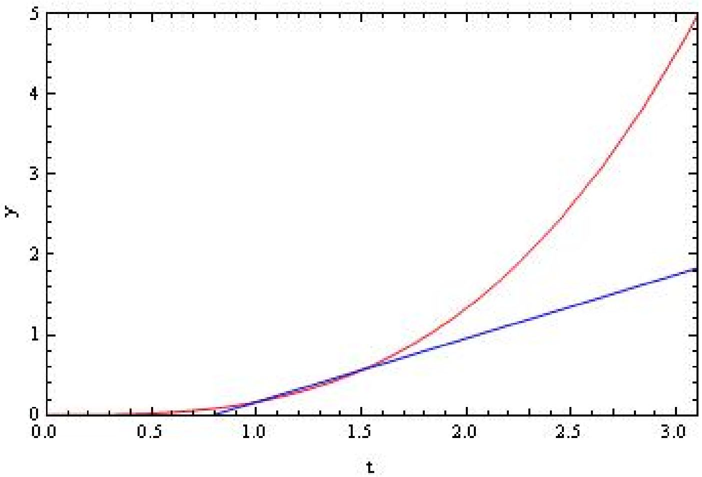
To get the plots that are asked for, we can use *Mathematica*, or just sketch them by hand. We don't really need to know the equation of the tangent line to sketch it, but we must be able to correctly construct the graphs. The intervals are listed below the graphs, and the secant drawn has a slope which is equal to the average velocity over that interval.



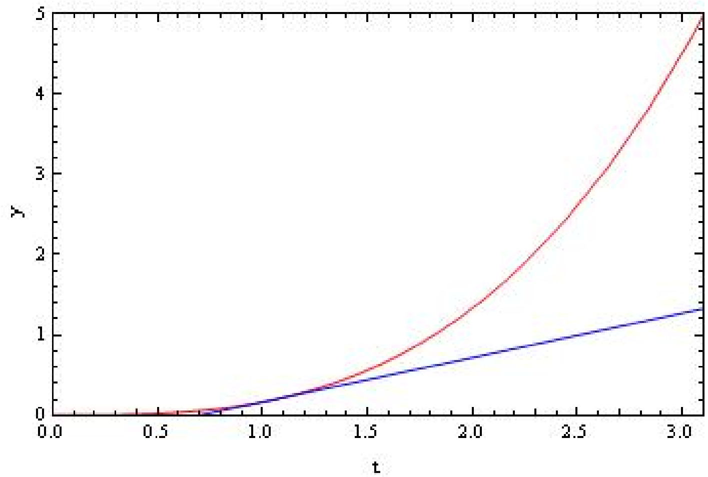
[1, 3]



[1, 2]

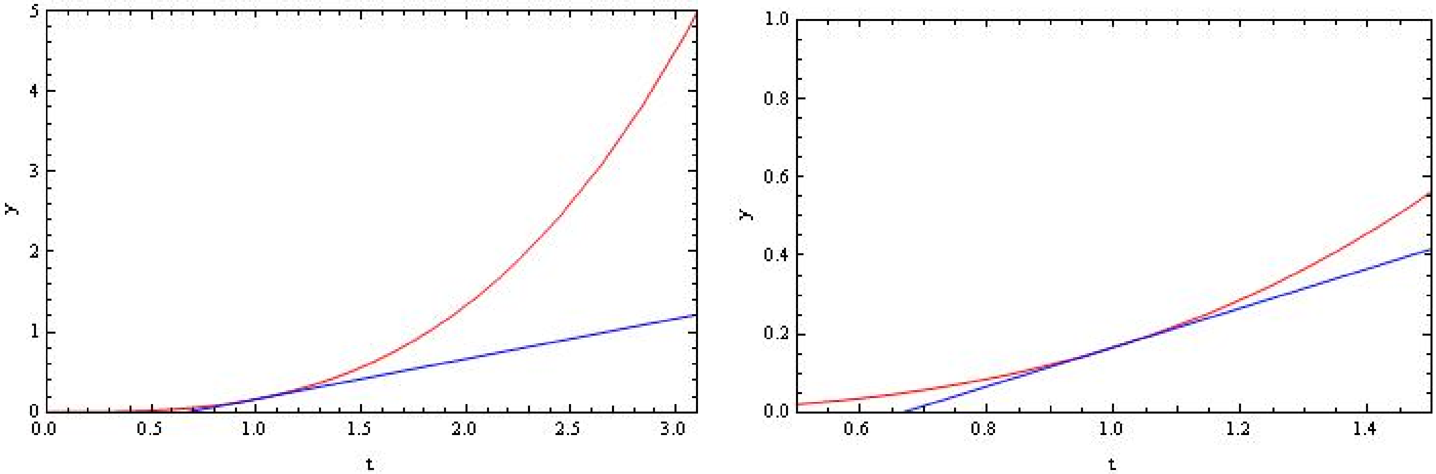


[1, 1.5]



[1, 1.1]

In the following graph, we have sketched the tangent line at the point where $t = 1$. This is the instantaneous velocity. The graph on the left is a zoom in on the graph on the right.



$[1, 1]$ (instantaneous velocity)