Questions

Example Differentiate the function $y = ae^v + \frac{b}{v} + \frac{c}{v^2}$.

Example Differentiate the function $y = A + \frac{B}{x} + \frac{C}{x^2}$.

Example Find an equation of the tangent line to the curve at the given point.

$y = x^4 + 2e^x, \quad (0, 2).$

Example Find an equation of the tangent line to the curve at the given point. Illustrate by graphing the curve and the tangent line on the same screen.

$y = x^2 + 2e^x, \quad (0, 2).$

Example Find a cubic function $y = ax^3 + bx^2 + cx + d$ whose graph has horizontal tangents at the points $(-2, 6)$ and $(2, 0)$.

Solutions

Example Differentiate the function $y = ae^v + \frac{b}{v} + \frac{c}{v^2}$.

To differentiate the function $y = f(v) = ae^v + \frac{b}{v} + \frac{c}{v^2}$ we first should rewrite it. You can find the derivative by other methods (quotient rule), but the method I present is the most direct. I am including what derivative rule was used, but you need not do that in your solution.

$$\begin{aligned}
y &= ae^v + \frac{b}{v} + \frac{c}{v^2} \\
&= ae^v + bv^{-1} + cv^{-2} \\
\frac{dy}{dv} &= \frac{d}{dv}[ae^v + bv^{-1} + cv^{-2}] \\
&= \frac{d}{dv}[ae^v] + \frac{d}{dv}[bv^{-1}] + \frac{d}{dv}[cv^{-2}] \quad \text{Sum Rule} \\
&= a\frac{d}{dv}[e^v] + b\frac{d}{dv}[v^{-1}] + c\frac{d}{dv}[v^{-2}] \quad \text{Constant Multiple Rule} \\
&= ae^v + b(-1)v^{-1-1} + c(-2)v^{-2-1} \quad \text{Exponential Rule; Power Rule} \\
&= ae^v - bv^{-2} - 2cv^{-3} \\
&= ae^v - \frac{b}{v^2} - 2\frac{c}{v^3}
\end{aligned}$$

Example Differentiate the function $y = A + \frac{B}{x} + \frac{C}{x^2}$.

To differentiate the function $y = A + \frac{B}{x} + \frac{C}{x^2}$ we first should rewrite it. You can find the derivative by other methods (quotient rule), but the method I present is the most direct. I am including what derivative rule was used, but you need not do that in your solution.

$$
\begin{aligned}
y &= A + \frac{B}{x} + \frac{C}{x^2} \\
&= A + Bx^{-1} + Cx^{-2} \\
\frac{dy}{dx} &= \frac{d}{dx}[A + Bx^{-1} + Cx^{-2}] \\
&= \frac{d}{dx}[A] + \frac{d}{dx}[Bx^{-1}] + \frac{d}{dx}[Cx^{-2}] \quad \text{Sum Rule} \\
&= [0] + B\frac{d}{dx}[x^{-1}] + C\frac{d}{dx}[x^{-2}] \quad \text{Constant Rule \& Constant Multiple Rule} \\
&= B(-1)x^{-1-1} + C(-2)x^{-2-1} \quad \text{Power Rule} \\
&= -Bx^{-2} - 2Cx^{-3} \\
&= -\frac{B}{x^2} - 2\frac{C}{x^3}
\end{aligned}
$$

Example Find an equation of the tangent line to the curve at the given point.

$y = x^4 + 2e^x, \quad (0, 2).$

I will answer this question by first writing out some statements that I will use to solve the process.

The derivative is equal to the slope of the tangent line.
Therefore, we want to find $f'(x)$.
The function is $f(x) = x^4 + 2e^x$.
The point we are interested in is $(0, 2)$, which means $x = 0$.
The slope of the tangent line at $x = 0$ is $f'(0)$.
The tangent line goes through the point $(0, 2)$.
The equation of the tangent line can be found from $y - y_0 = m(x - x_0)$.
The equation of the tangent line will be $y - 2 = f'(0)(x - 0)$.

$$
\begin{aligned}
f(x) &= x^4 + 2e^x \\
f'(x) &= \frac{d}{dx}[x^4 + 2e^x] \\
&= 4x^3 + 2e^x \quad \text{Power Rule, Exponential Rule} \\
f'(0) &= 4(0)^3 + 2e^0 = 2
\end{aligned}
$$

The equation of the tangent line is therefore:

$$
\begin{aligned}
y - 2 &= 2(x - 0) \\
y &= 2x + 2
\end{aligned}
$$

Example Find an equation of the tangent line to the curve at the given point. Illustrate by graphing the curve and the tangent line on the same screen.

$y = x^2 + 2e^x, \quad (0, 2).$

I will answer this question by first writing out some statements that I will use to solve the process.

The derivative is equal to the slope of the tangent line.
Therefore, we want to find $f'(x)$.
The function is $f(x) = x^2 + 2e^x$.
The point we are interested in is $(0, 2)$, which means $x = 0$.
The slope of the tangent line at $x = 0$ is $f'(0)$.
The tangent line goes through the point $(0, 2)$.
The equation of the tangent line can be found from $y - y_0 = m(x - x_0)$.
The equation of the tangent line will be $y - 2 = f'(0)(x - 0)$.

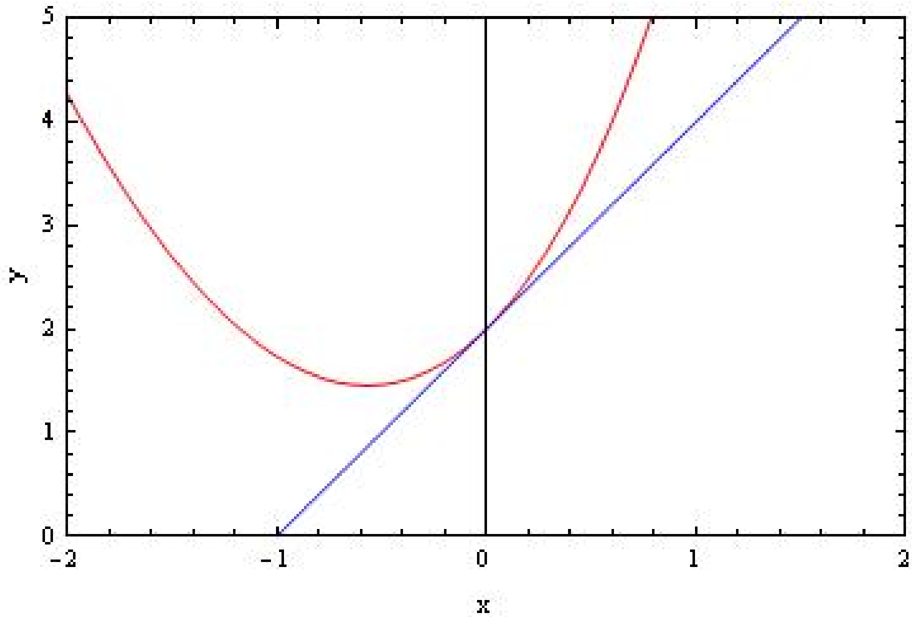
$$\begin{aligned} f(x) &= x^2 + 2e^x \\ f'(x) &= \frac{d}{dx}[x^2 + 2e^x] \\ &= 2x + 2e^x \quad \text{Power Rule, Exponential Rule} \\ f'(0) &= 2(0) + 2e^0 = 2 \end{aligned}$$

The equation of the tangent line is therefore:

$$\begin{aligned} y - 2 &= 2(x - 0) \\ y &= 2x + 2 \end{aligned}$$

Here is a picture of the situation. The *Mathematica* commands I used were are included, but you probably wouldn't have used all the options I used for the graph.

```
f[x_] = x^2 + 2 Exp[x]
ytangent[x_] =  2 x + 2
plot1 = Plot[{f[x], ytangent[x]}, {x, -2, 2}, PlotRange -> {{-2, 2}, {0, 5}},
    Frame -> True, FrameLabel -> {"x", "y"},
    PlotStyle -> {Red, Blue}]
```



Example Find a cubic function $y = ax^3 + bx^2 + cx + d$ whose graph has horizontal tangents at the points $(-2, 6)$ and $(2, 0)$.

This is a fun question! Here are some statements that help me solve the problem. First, let $f(x) = ax^3 + bx^2 + cx + d$.

We need to determine the values for the constants a, b, c, d.
To determine the value of four constants, we will need four equations.
The problem tells us that the curve must pass through the points $(-2, 6)$ and $(2, 0)$.
One equation will therefore be $f(-2) = 6 \longrightarrow a(-2)^3 + b(-2)^2 + c(-2) + d = 6 \longrightarrow -8a + 4b - 2c + d = 6$.
A second equation will therefore be $f(2) = 0 \longrightarrow a(2)^3 + b(2)^2 + c(2) + d = 0 \longrightarrow 8a + 4b + 2c + d = 0$.

The derivative is equal to the slope of the tangent line.
Therefore, we want to find $f'(x)$.
$f'(x) = 3ax^2 + 2bx + c$.
If the tangent line is horizontal, $f'(x) = 0$.
The points we are interested in are $(-2, 6)$, which means $x = -2$, and $(2, 0)$ which means $x = 2$.
A third equation will be $f'(-2) = 0 \longrightarrow 3a(-2)^2 + 2b(-2) + c = 0 \longrightarrow 12a - 4b + c = 0$.
A fourth equation will be $f'(2) = 0 \longrightarrow 3a(2)^2 + 2b(2) + c = 0 \longrightarrow 12a + 4b + c = 0$.

So the problem comes down to solving the system of four equation in four unknowns a, b, c, d:

$$\begin{aligned} -8a + 4b - 2c + d &= 6 \\ 8a + 4b + 2c + d &= 0 \\ 12a - 4b + c &= 0 \\ 12a + 4b + c &= 0 \end{aligned}$$

You will learn very cool methods of solving systems of equations like this in linear algebra (Cramer's rule). For now, you could solve one equation for a, and then substitute into another, etc. Or, we can have *Mathematica* do it for us.

```
eq1 = -8 a + 4b - 2c + d == 6
eq2 = 8 a + 4b + 2c + d == 0
eq3 = 12 a - 4 b + c == 0
eq4 = 12 a + 4 b + c == 0
Solve[{eq1, eq2, eq3, eq4}, {a, b, c, d}]
```

which gives us the solution $a = 3/16, b = 0, c = -9/4, d = 3$, and so the cubic function with the desired properties is

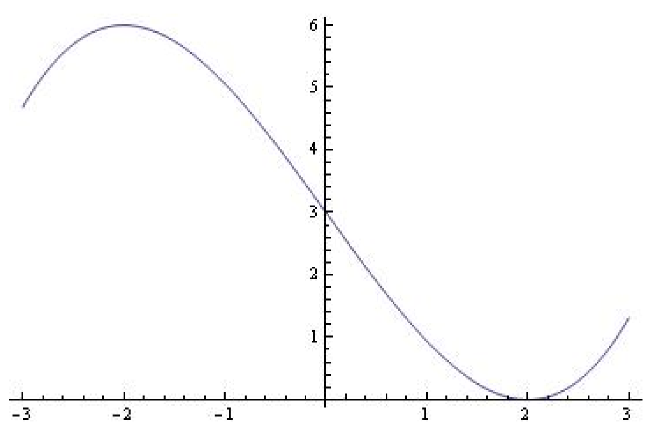
$$y = \frac{3}{16}x^3 - \frac{9}{4}x + 3.$$

Since we used *Mathematica* to solve the system of equations, and we maybe aren't sure that that was done correctly, we should check our solution. We can check it using the following *Mathematica* commands:

```
f[x_] = 3 x^3/16 - 9x/4 + 3
f[-2]
f[2]
f'[-2]
f'[2]
```

The *Mathematica* output tells us that we have the correct function. A simple plot would also verify we had the correct function:

```
Plot[f[x], {x, -3, 3}]
```



Notice the plot passes through the points $(-2, 6)$ and $(2, 0)$, and the tangent line is horizontal at those points.