Questions

Example Given $xy + 2x + 3x^2 = 4$.

a) Find y' by implicit differentiation.

b) Solve the equation explicitly for y and differentiate to get $y\prime$ explicitly in terms of x.

c) Verify the solutions in part a) and b) are the same.

Example Find dy/dx by implicit differentiation, $x^2y + xy^2 = 3x$.

Example Graph the curve with equation $y(y^2 - 1)(y - 2) = x(x - 1)(x - 2)$. At how many points does this curve have horizontal tangents? Estimate the x-coordinates of these points. Find equations of the tangents lines at the points $(0, 1)$ and $(0, 2)$. Find the exact x-coordinates of the points in part a).

Solutions

Example Given $xy + 2x + 3x^2 = 4$.

a) Find y' by implicit differentiation.

b) Solve the equation explicitly for y and differentiate to get $y\prime$ explicitly in terms of x.

c) Verify the solutions in part a) and b) are the same.

First, get the derivative implicitly. We think of y as a function of x when taking the derivative.

$$\begin{aligned}
\frac{d}{dx}[xy + 2x + 3x^2 &= 4] \\
\frac{d}{dx}[xy] + 2\frac{d}{dx}[x] + 3\frac{d}{dx}[x^2] &= \frac{d}{dx}[4] \\
y\frac{d}{dx}[x] + x\frac{d}{dx}[y] + 2(1) + 3(2x) &= 0 \\
y(1) + x\frac{dy}{dx} + 2 + 6x &= 0 \\
\frac{dy}{dx} &= \frac{-2 - 6x - y}{x} \qquad (1)
\end{aligned}$$

This is an implicit form for dy/dx, which is an acceptable way to leave the solution since the original equation was implicit.

Now, let's solve for y to get an explicit equation, and get the derivative explicitly.

$$\begin{aligned}
xy + 2x + 3x^2 &= 4 \text{ (implicit function)} \\
y &= \frac{4 - 2x - 3x^2}{x} \text{ (explicit function)} \qquad (2) \\
\frac{dy}{dx} &= \frac{d}{dx}\left[\frac{4 - 2x - 3x^2}{x}\right] \\
&= \frac{x\frac{d}{dx}(4 - 2x - 3x^2) - (4 - 2x - 3x^2)\frac{d}{dx}[x]}{x^2} \\
&= \frac{x(-2 - 6x) - (4 - 2x - 3x^2)(1)}{x^2} \\
&= \frac{-2x - 6x^2 - 4 + 2x + 3x^2}{x^2} \\
&= \frac{-3x^2 - 4}{x^2}
\end{aligned}$$

To show these are the same, we can substitute Eq. (2) into Eq. (1).

$$\begin{aligned}
\frac{dy}{dx} &= \frac{-2 - 6x - y}{x} \\
&= \frac{-2 - 6x - \left(\frac{4 - 2x - 3x^2}{x}\right)}{x} \\
&= \frac{\left(\frac{-2x - 6x^2 - 4 + 2x + 3x^2}{x}\right)}{x} = \frac{\left(\frac{-3x^2 - 4}{x}\right)}{x} = \frac{-3x^2 - 4}{x^2}
\end{aligned}$$

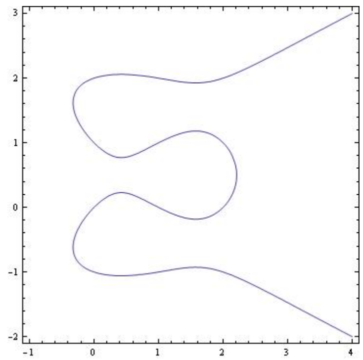
Example Find dy/dx by implicit differentiation, $x^2y + xy^2 = 3x$.

$$\begin{aligned}
&x^2y + xy^2 = 3x \\
&\frac{d}{dx}[x^2y + xy^2 = 3x] \\
&\frac{d}{dx}[x^2y] + \frac{d}{dx}[xy^2] = \frac{d}{dx}[3x] \\
&x^2\frac{d}{dx}[y] + y\frac{d}{dx}[x^2] + x\frac{d}{dx}[y^2] + y^2\frac{d}{dx}[x] = 3 \\
&x^2\frac{dy}{dx} + y(2x) + x(2y\frac{dy}{dx}) + y^2 = 3 \\
&\frac{dy}{dx}(x^2 + 2xy) + 2xy + y^2 = 3 \\
&\frac{dy}{dx} = \frac{3 - 2xy - y^2}{x^2 + 2xy}
\end{aligned}$$

This is an implicit formula for the derivative.

Example Graph the curve with equation $y(y^2 - 1)(y - 2) = x(x - 1)(x - 2)$. At how many points does this curve have horizontal tangents? Estimate the x-coordinates of these points. Find equations of the tangents lines at the points $(0, 1)$ and $(0, 2)$. Find the exact x-coordinates of the points in part a).

```
ContourPlot[y(y^2 - 1)(y - 2) == x(x - 1)(x - 2), {x, -1, 4}, {y, -2, 3}]
```



From the graph, we see the figure has horizontal tangents at eight separate points. A rough guess would be that these horizontal tangents occur at $x = 0.5$ and $= 1.5$.

The tangent line has equation $y - y_0 = m(x - x_0)$ where (x_0, y_0) is a point on the tangent line and m is the slope of the tangent, i.e., $m = dy/dx$ evaluated at the point where the tangent line touches the curve.

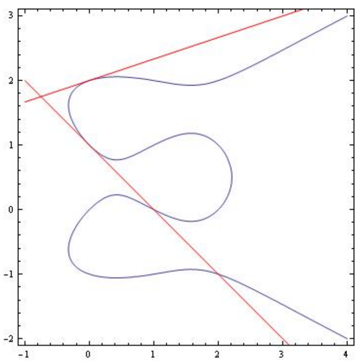
So we need a derivative,

$$
\begin{aligned}
&y(y^2-1)(y-2) = x(x-1)(x-2)\\
&2y - y^2 - 2y^3 + y^4 = 2x - 3x^2 + x^3 \text{ (expand to make differentiating easier)}\\
&\frac{d}{dx}[2y - y^2 - 2y^3 + y^4 = 2x - 3x^2 + x^3] \text{ (implicitly differentiate)}\\
&\frac{d}{dx}[2y] - \frac{d}{dx}[y^2] - 2\frac{d}{dx}[y^3] + \frac{d}{dx}[y^4] = \frac{d}{dx}[2x - 3x^2 + x^3]\\
&2\frac{dy}{dx} - 2y\frac{dy}{dx} - 6y^2\frac{dy}{dx} + 4y^3\frac{dy}{dx} = 2 - 6x + 3x^2\\
&\frac{dy}{dx}(2 - 2y - 6y^2 + 4y^3) = 2 - 6x + 3x^2\\
&\frac{dy}{dx} = \frac{2 - 6x + 3x^2}{2 - 2y - 6y^2 + 4y^3}
\end{aligned}
$$

At $(0,1)$, $\left.\frac{dy}{dx}\right|_{(0,1)} = \frac{2 - 6(0) + 3(0)^2}{2 - 2(1) - 6(1)^2 + 4(1)^3} = -1.$
The equation of the tangent line at $(0,1)$ is $y - 1 = -1(x - 0)$, or $y = -x + 1$.

At $(0,2)$, $\left.\frac{dy}{dx}\right|_{(0,2)} = \frac{2 - 6(0) + 3(0)^2}{2 - 2(2) - 6(2)^2 + 4(2)^3} = \frac{1}{3}.$
The equation of the tangent line at $(0,1)$ is $y - 2 = \frac{1}{3}(x - 0)$, or $y = x/3 + 2$.

A plot of the tangent lines looks like the following:



The exact x coordinates of the horizontal tangents occur when the $dy/dx = 0$, which occurs when $2 - 6x + 3x^2 = 0$. Using the quadratic equation to solve this we find $x = 1 \pm \frac{\sqrt{3}}{3}$. The decimal value of these roots is 0.42265 and 1.57735, so our guesses in part a) were not far off.

My fun curve is the following, which has lots of horizontal tangents.

```
ContourPlot[
  Sin[y](y^2 - 1)(x Cos[y^2] - 2) == Cos[x^2](x - 1)(Log[Abs[x^5]] - 2), {x, -7, 7}, {y, -7, 7},
  AxesLabel -> {"x", "y"}, PlotPoints -> 40, ImageSize -> {600, Automatic}]
```