Questions

Example Differentiate the function $f(t) = \dfrac{1+\ln t}{1-\ln t}$.

Example Differentiate the function $g(x) = \ln \dfrac{a-x}{a+x}$.

Example Use logarithmic differentiation to find the derivative of $y = x^{1/x}$.

Solutions

Example Differentiate the function $f(t) = \dfrac{1+\ln t}{1-\ln t}$.

$$
\begin{aligned}
f(t) &= \frac{1+\ln t}{1-\ln t} \\
f'(t) &= \frac{d}{dt}\left[\frac{1+\ln t}{1-\ln t}\right] \\
&= \frac{(1-\ln t)\frac{d}{dt}[(1+\ln t)] - (1+\ln t)\frac{d}{dt}[(1-\ln t)]}{(1-\ln t)^2} \\
&= \frac{(1-\ln t)\left(\frac{1}{t}\right) - (1+\ln t)\left(\frac{-1}{t}\right)}{(1-\ln t)^2} \\
&= \frac{(1-\ln t)\left(\frac{1}{t}\right) + (1+\ln t)\left(\frac{1}{t}\right)}{(1-\ln t)^2} \\
&= \frac{\left(\frac{2}{t}\right)}{(1-\ln t)^2} \\
&= \frac{2}{t(1-\ln t)^2}
\end{aligned}
$$

Example Differentiate the function $g(x) = \ln \dfrac{a-x}{a+x}$.

$$
\begin{aligned}
g(x) &= \ln \frac{a-x}{a+x} \\
\frac{d}{dx} g(x) &= \frac{d}{dx}\left[\ln \frac{a-x}{a+x}\right] \\
&= \frac{d}{dx}\left[\ln u\right], u = \frac{a-x}{a+x} \\
&= \frac{d}{du}\left[\ln u\right] \cdot \frac{du}{dx}, u = \frac{a-x}{a+x} \quad \text{(chain rule)}
\end{aligned}
$$

$$
\begin{aligned}
\frac{d}{du}\left[\ln u\right] &= \frac{1}{u} = \frac{a+x}{a-x} \\
\frac{du}{dx} &= \frac{d}{dx}[\frac{a-x}{a+x}] \\
&= \frac{(a+x)\frac{d}{dx}[(a-x)] - (a-x)\frac{d}{dx}[(a+x)]}{(a+x)^2} \\
&= \frac{(a+x)(-1) - (a-x)(1)}{(a+x)^2} \\
&= \frac{-a-x-a+x}{(a+x)^2} \\
&= \frac{-2a}{(a+x)^2}
\end{aligned}
$$

Now, collect everything together:

$$
\begin{aligned}
\frac{d}{dx} g(x) &= \frac{d}{du}\left[\ln u\right] \cdot \frac{du}{dx} \\
&= \frac{a+x}{a-x} \cdot \frac{-2a}{(a+x)^2} \\
&= \frac{-2a}{(a-x)(a+x)}
\end{aligned}
$$

Example Use logarithmic differentiation to find the derivative of $y = x^{1/x}$.

$$\begin{aligned}
y &= x^{1/x} \\
\ln[y &= x^{1/x}] \text{ (take logarithm of whole equation)} \\
\ln y &= \ln[x^{1/x}] \text{ (use log laws to simplify)} \\
\ln y &= \frac{\ln x}{x} \text{ (implicitly differentiate)} \\
\frac{d}{dx}\ln y &= \frac{d}{dx}\left(\frac{\ln x}{x}\right) \\
\frac{1}{y}\frac{dy}{dx} &= \frac{x\frac{d}{dx}[\ln x] - \ln x\frac{d}{dx}[x]}{x^2} \\
\frac{dy}{dx} &= y\left(\frac{x\left(\frac{1}{x}\right) - \ln x(1)}{x^2}\right) \\
&= x^{1/x}\left(\frac{1-\ln x}{x^2}\right)
\end{aligned}$$