Related Rates in General

Related rates means *related rates of change*, and since rates of changes are derivatives, related rates really means *related derivatives.*

The only way to learn how to solve related rates problems is to practice.

The procedure to solve a related rates problem:

1. Write down the rate which is Given.
2. Write down the rate which is Unknown.
3. Write down your notation and draw a diagram.
4. Find a formula connecting the the quantities you listed in your Notation. There should be no derivatives in this relationship.
   (a) If necessary, use geometry to eliminate a variable from your formula.
5. Implicitly differentiate the formula to get rates of change involved. If you end up with more than one unknown rate of change, you might have to eliminate a variable using geometry (as mentioned in the previous step).
6. Solve for the Unknown Rate.
7. Substitute values to determine the Unknown Rate.
8. Write a concluding sentence.

Questions

1. A man starts walking north 4 ft/s from a point P. Five minutes later a woman starts walking south at 5 ft/s from a point 500 ft due east of P. At what rate are the people moving apart 15 min after the woman starts walking?

2. The altitude of a triangle is increasing at a rate of 1 cm.min while the area of the triangle is increasing at a rate of 2 cm^2/min. At what rate is the base of the triangle changing when the altitude is 10 cm and the area is 100 cm^2?

3. Water is leaking out of an inverted conical tank at a rate of 10000 cm^3/min at the same time that water is being pumped into the tank at a constant rate. The tank has height 6 m and diameter at the top is 4 m. If the water level is rising at a rate of 20 cm/min when the height of the water is 2 m, find the rate at which water is being pumped into the tank.

4. A baseball diamond is a square with side 90 ft. A batter hits the ball and runs toward first base with a speed of 24 ft/s.

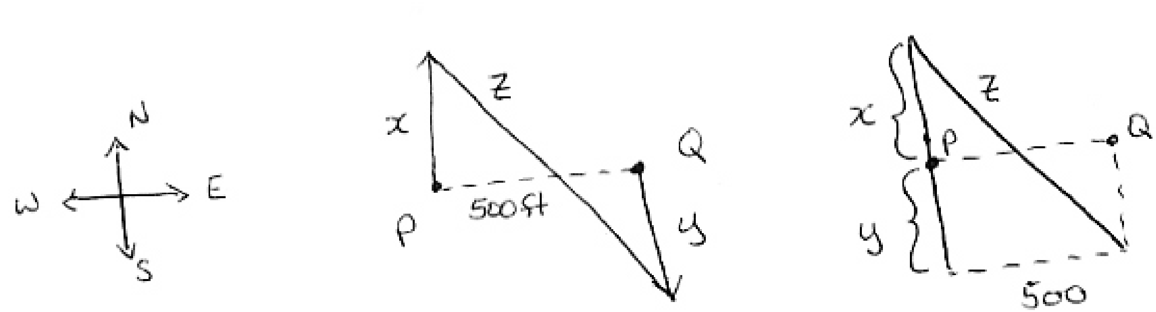
- At what rate is his distance from second base decreasing when he is halfway to first base?
- At what rate is his distance from third base increasing at the same moment?

5. A trough is 10 ft long and its ends have the shape of isosceles triangles that are 3 ft across at the top and have a height of 1 ft. If the trough is being filled with water at the rate of 12 ft^3/min, how fast is the water rising when the water is 6 inches deep?

Solutions

1. A man starts walking north 4 ft/s from a point P. Five minutes later a woman starts walking south at 5 ft/s from a point 500 ft due east of P. At what rate are the people moving apart 15 min after the woman starts walking?

Here is a diagram of the situation:



The notation I have introduced is:
Distance man is from P is x.
Distance woman is from Q is y.
Distance between them is z.

We are given: The man walks with speed 4 ft/s. This means $\left|\frac{dx}{dt}\right| = 4$ ft/s $= 240$ ft/min.
The woman walks with speed 5 ft/s. This means $\left|\frac{dy}{dt}\right| = 5$ ft/s $= 300$ ft/min.
The distance between P and Q is 500 ft.

The units have been changed to ensure they are consistent, in feet and minutes.

What is unknown is the rate at which the are moving apart, which is the rate of change of the distance between them, $\left|\frac{dz}{dt}\right|$.

To get the relation between x, y, and z we need to use our diagram. It is easier to see the relation if we redraw our diagram, which I already did above. The relation is

$$(x+y)^2 + 500^2 = z^2.$$

Implicitly differentiate the relation to get a relation between the rates of change. The rates of change are with respect to time t, so we should differentiate with respect to t. The quantities x, y, and z are all functions of t.

$$\begin{aligned}\frac{d}{dt}[z^2 &= (x+y)^2 + 500^2] \\ 2z\frac{dz}{dt} &= 2(x+y)\left(\frac{dx}{dt} + \frac{dy}{dt}\right)\end{aligned}$$

We solve this for the unknown rate of change:

$$\frac{dz}{dt} = \frac{(x+y)}{z}\left(\frac{dx}{dt} + \frac{dy}{dt}\right)$$

To use this equation, we need to know the quantities x, y, z after the woman has been walking for 15 minutes. Since she started walking 5 minutes after the man, the man will have been walking for 20 minutes.

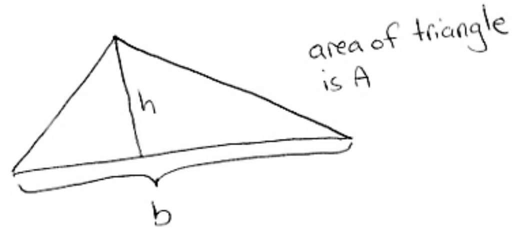
In 15 minutes, the woman walks $y = 15$ min $\cdot$ 300 ft/min= 4500 ft.
In 20 minutes, the man walks $x = 20$ min $\cdot$ 240 ft/min= 4800 ft.
The distance between them at this time will be $z = \sqrt{(x+y)^2 + 500^2} = \sqrt{(4800+4500)^2 + 500^2} = 100\sqrt{8674}$ ft.

The rate of change of the distance between them after the woman has been walking 15 minutes is

$$\frac{dz}{dt} = \frac{(x+y)}{z}\left(\frac{dx}{dt}+\frac{dy}{dt}\right) = \frac{(4800+4500)}{100\sqrt{8674}}(4+5) = \frac{837}{\sqrt{8674}}\text{ft/min.}$$

2. The altitude of a triangle is increasing at a rate of 1 cm/min while the area of the triangle is increasing at a rate of 2 cm^2/min. At what rate is the base of the triangle changing when the altitude is 10 cm and the area is 100 cm^2?

Here is a diagram of the situation:



The notation I have introduced is:
The altitude of the triangle is h.
The base of the triangle is b.
The area of the triangle is A.

We are given:
The altitude is increasing at a rate of $\frac{dh}{dt} = 1$ cm/min.
The area is increasing at a rate of $\frac{dA}{dt} = 2$ cm^2/min.

What is unknown is the rate of change of the base, $\frac{db}{dt}$.

The relation between the base and altitude of a triangle is

$$A = \frac{1}{2}bh.$$

Implicitly differentiate with respect to time:

$$\frac{dA}{dt} = \frac{1}{2}\left(h\frac{db}{dt}+b\frac{dh}{dt}\right).$$

Solve for the unknown rate of change:

$$\frac{db}{dt} = \frac{1}{h}\left(2\frac{dA}{dt}-b\frac{dh}{dt}\right).$$

At $h = 10$ cm and $A = 100$ cm^2, $b = 2A/h = 2(100)/10 = 20$ cm.

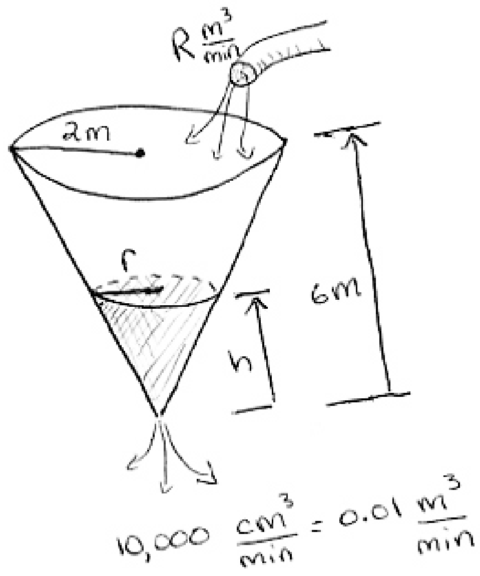
The rate of change of the base at this time is

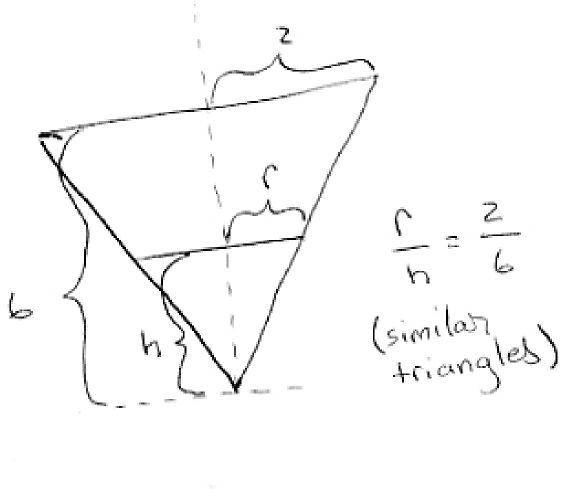
$$\frac{db}{dt} = \frac{1}{10}\left(2(2)-(20)(1)\right) = -1.6 \text{ cm/min.}$$

The negative sign in our answer means the length of the base is decreasing.

3. Water is leaking out of an inverted conical tank at a rate of 10000 cm^3/min at the same time that water is being pumped into the tank at a constant rate. The tank has height 6 m and diameter at the top is 4 m. If the water level is rising at a rate of 20 cm/min when the height of the water is 2 m, find the rate at which water is being pumped into the tank.

Here is a diagram of the situation:





The notation I have introduced is:
The height of water in the tank is h m.
The radius of water in the tank is r m.
The rate water is being pumped into the tank is R m^3/min.

We are given:
Water is leaking out of the tank at a rate of $= 10\,000\ \text{cm}^3/\text{min} = 10^4\ (10^{-2}\ \text{m})^3/\text{min} = = 0.01\ \text{m}^3/\text{min}$.
The tank has height 6 m and radius at top of 2 m.

What is unknown is the rate water is being pumped in, R.

The volume of water in the tank at a specific time is given by

$$V = \frac{1}{3}\pi r^2 h.$$

We can eliminate one of the variables using similar triangles.

$$\frac{r}{h} = \frac{2}{6} \longrightarrow r = \frac{1}{3}h.$$

The volume of water in the tank is given by

$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi \left(\frac{1}{3}h\right)^2 h = \frac{1}{27}\pi h^3.$$

This is the volume for a conical tank of the specific dimensions given in this problem.

The rate of change of volume of water in the tank is found by implicitly differentiating:

$$\frac{dV}{dt} = \frac{1}{9}\pi h^2 \frac{dh}{dt}\ \text{m/min}$$

which must equal

$$R - 0.01\ \text{m/min}$$

At $h = 2$ m, $\dfrac{dh}{dt} = 20$ cm/min $= 0.2$ m/min, and we have

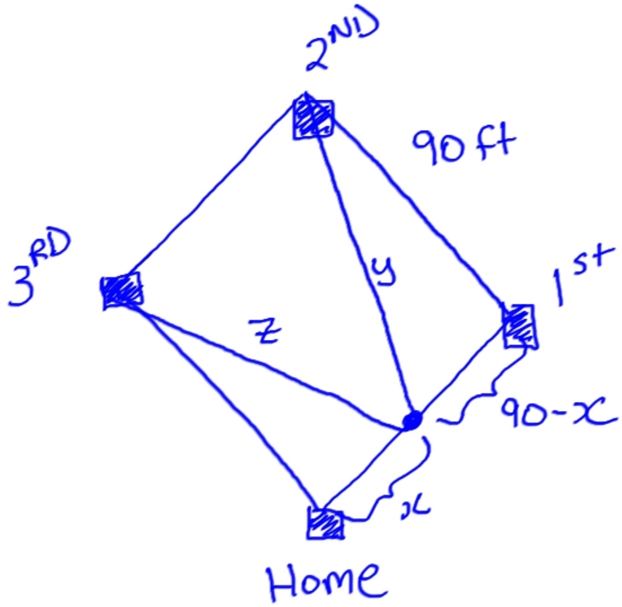
$$\begin{aligned} R - 0.01 &= \frac{1}{9}\pi h^2 \frac{dh}{dt} \\ R &= \frac{1}{9}\pi h^2 \frac{dh}{dt} + 0.01 \\ &= \frac{1}{9}\pi (2)^2 (0.2) + 0.01 \\ &= \frac{1}{9}\pi (2)^2 (0.2) + 0.01 \\ &= 0.289 \text{m}^3/\text{min} \end{aligned}$$

The rate water is being pumped into the tank is 0.289 m^3/min when the height of the water is 2m.

4. A baseball diamond is a square with side 90 ft. A batter hits the ball and runs toward first base with a speed of 24 ft/s.

- At what rate is his distance from second base decreasing when he is halfway to first base?
- At what rate is his distance from third base increasing at the same moment?

Diagram:



x is the distance from home plate.
$90 - x$ is the distance from the runner to first base.
y is the distance from runner to second base.
z is the distance from runner to third base.
Given the rate of change of distance from home 24 ft/s $= \frac{dx}{dt}$. Need to find $\frac{dy}{dt}$.

Relation from using Pythagorean Theorem: $y^2 = 90^2 + (90 - x)^2$.

Implicitly differentiate and solve for $\frac{dy}{dt}$:

$$\begin{aligned}
\frac{d}{dt}[y^2] &= \frac{d}{dt}[90^2 + (90-x)^2] \\
\frac{d}{dy}[y^2]\frac{dy}{dt} &= \frac{d}{dx}[(90-x)^2]\frac{dx}{dt} \\
2y\frac{dy}{dt} &= 2(90-x)(-1)\frac{dx}{dt} \\
\frac{dy}{dt} &= -\frac{(90-x)}{y}\frac{dx}{dt} \\
&= -\frac{(90-x)}{\sqrt{90^2+(90-x)^2}}\frac{dx}{dt}
\end{aligned}$$

When the runner is halfway between first and home, $x = 45$ ft and $\frac{dx}{dt} = 24$ ft/s, so substituting this in we get

$$\frac{dy}{dt} = -\frac{24}{\sqrt{5}} \text{ ft/s.}$$

The answer is negative since the distance to second base is decreasing.

A similar process for the distance to third looks like:

Relation from using Pythagorean Theorem: $z^2 = 90^2 + x^2$.

Implicitly differentiate and solve for $\frac{dz}{dt}$:

$$\begin{aligned}
\frac{d}{dt}[z^2] &= \frac{d}{dt}[90^2 + x^2] \\
\frac{d}{dz}[z^2]\frac{dz}{dt} &= \frac{d}{dx}[x^2]\frac{dx}{dt} \\
2z\frac{dz}{dt} &= 2x\frac{dx}{dt} \\
\frac{dy}{dt} &= \frac{x}{z}\frac{dx}{dt} \\
&= \frac{x}{\sqrt{90^2+x^2}}\frac{dx}{dt}
\end{aligned}$$

When the runner is halfway between first and home, $x = 45$ ft and $\frac{dx}{dt} = 24$ ft/s, so substituting this in we get

$$\frac{dz}{dt} = \frac{24}{\sqrt{5}} \text{ ft/s.}$$

The answer is positive since the distance to third base is increasing.

5. A trough is 10 ft long and its ends have the shape of isosceles triangles that are 3 ft across at the top and have a height of 1 ft. If the trough is being filled with water at the rate of 12 ft^3/min, how fast is the water rising when the water is 6 inches deep?

Diagram:

We are given the rate of change of the volume is $\frac{dV}{dt} = 12$ ft^3/min.

The unknown rate of change is the water level, $\frac{dh}{dt}$.

The volume is $V = \frac{1}{2}bh10 = 5bh$.

We need to eliminate the variable b, since we know nothing about it. Use similar triangles to get

$$\frac{b}{3} = \frac{h}{1} \to b = 3h.$$

No we can substitute this into our volume equation and implicitly differentiate:

$$\begin{aligned} V &= 5bh \\ V &= 5(3h)h \\ V &= 15h^2 \\ \frac{dV}{dt} &= \frac{d}{dt}[15h^2] \\ &= 15\frac{d}{dh}[h^2]\frac{dh}{dt} \\ &= 30h\frac{dh}{dt} \\ \frac{dh}{dt} &= \frac{1}{30h}\frac{dV}{dt} \end{aligned}$$

When the water level is $h = 6$ inches $= 1/2$ ft deep, the water is rising at the rate of

$$\frac{dh}{dt} = \frac{1}{30(1/2)}(12) = \frac{4}{5} \text{ ft/min.}$$