Questions

Example Find the absolute maximum and absolute minimum values of $f(x) = \dfrac{x}{x^2+1}$ on the interval $[0, 2]$

Example Find the absolute maximum and absolute minimum values of $f(x) = \dfrac{\ln x}{x}$ on the interval $[1, 3]$

Example If a and b are positive numbers, find the maximum value of $f(x) = x^a(1-x)^b,\ \ 0 \le x \le 1$.

Solutions

Example Find the absolute maximum and absolute minimum values of $f(x) = \dfrac{x}{x^2+1}$ on the interval $[0, 2]$

Absolute extrema on a closed interval are found using the Closed Interval Method:
1) Find the values of f at the critical numbers of f in (a, b).
2) Find the values of f at the endpoints of the interval.
3) The largest of the values from 1) and 2) is the absolute maximum; the smallest of these values is the absolute minimum.

We need the critical numbers. We need to find where $f'(c) = 0$ and where $f'(x)$ does not exist. Since $x^2 + 1 \neq 0$ for real valued x, the derivative always exists.

$$\begin{aligned}
f(x) &= \frac{x}{x^2+1} \\
f'(x) &= \frac{d}{dx}\left[\frac{x}{x^2+1}\right] \\
&= \frac{(x^2+1)\frac{d}{dx}[x] - x\frac{d}{dx}[x^2+1]}{(x^2+1)^2} \\
&= \frac{(x^2+1)(1) - x(2x)}{(x^2+1)^2} \\
&= \frac{x^2+1-2x^2}{(x^2+1)^2} \\
&= \frac{-x^2+1}{(x^2+1)^2}
\end{aligned}$$

For $f'(c) = 0$, the numerator must equal zero,

$$\begin{aligned}
f'(c) = 0 &= \frac{-c^2+1}{(c^2+1)^2} \\
0 &= -c^2+1 \\
c^2 &= 1 \\
c &= \pm 1
\end{aligned}$$

We have shown that $f'(1) = 0$ and $f'(-1) = 0$. The critical numbers are $+1, -1$. Only $+1$ is in the interval $(0, 2)$.

Now, we evaluate the function at the critical numbers in the interval and at the endpoints of the interval:

$$\begin{aligned} f(+1) &= \frac{1}{(1)^2+1} = \frac{1}{2} \\ f(0) &= \frac{0}{(0)^2+1} = 0 \\ f(2) &= \frac{2}{(2)^2+1} = \frac{2}{5} \end{aligned}$$

The largest number is 1/2, so this is the absolute max and it occurs at $x = +1$. The smallest number is 0, so this is the absolute min and it occurs at $x = 0$.

Example Find the absolute maximum and absolute minimum values of $f(x) = \dfrac{\ln x}{x}$ on the interval $[1, 3]$

Absolute extrema on a closed interval are found using the Closed Interval Method:
1) Find the values of f at the critical numbers of f in (a, b).
2) Find the values of f at the endpoints of the interval.
3) The largest of the values from 1) and 2) is the absolute maximum; the smallest of these values is the absolute minimum.

We need the critical numbers, which means we need to find where $f'(c) = 0$ and where $f'(x)$ does not exist. The only place we could have the derivative not defined is for $x \leq 0$; luckily, this is outside of the interval $(1, 3)$ so we don't need to worry about the derivative being undefined.

$$\begin{aligned} f(x) &= \frac{\ln x}{x} \\ f'(x) &= \frac{d}{dx}\left[\frac{\ln x}{x}\right] \\ &= \frac{(x)\frac{d}{dx}[\ln x] - \ln x \frac{d}{dx}[x]}{(x)^2} \\ &= \frac{(x)\left(\frac{1}{x}\right) - \ln x(1)}{x^2} \\ &= \frac{1 - \ln x}{x^2} \end{aligned}$$

For $f'(c) = 0$, the numerator must equal zero,

$$\begin{aligned} f'(c) = 0 &= \frac{1 - \ln c}{c^2} \\ 0 &= 1 - \ln c \\ \ln c &= 1 \\ c &= e \end{aligned}$$

We have shown that $f'(e) = 0$. The critical number is e, which lies in the interval $(2, 3)$.

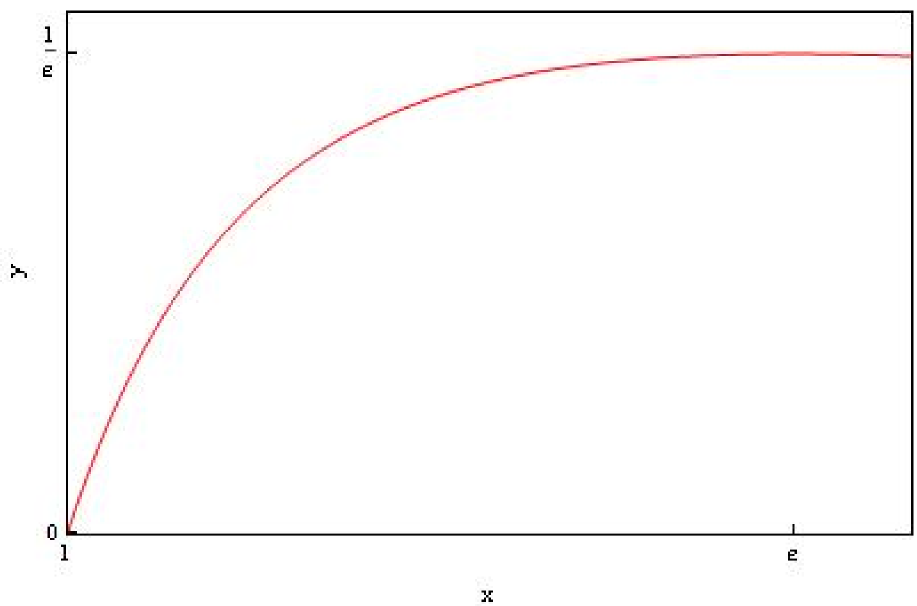
Now, we evaluate the function at the critical numbers in the interval and at the endpoints of the interval:

$$\begin{aligned} f(e) &= \frac{\ln e}{e} = \frac{1}{e} \\ f(1) &= \frac{\ln 1}{1} = 0 \\ f(3) &= \frac{\ln 3}{3} \end{aligned}$$

The smallest number is 0, so this is the absolute min and it occurs at $x = 1$.

It is difficult to determine if $1/e > \ln 3/3$ without resorting to a calculator, or more powerful techniques we have yet to learn. But we can argue based on the properties of derivatives that we *must* have $1/e > \ln 3/3$.

Since the function is continuous in the interval and has a minimum at $x = 1$, and the derivative at $x = e$ is zero means the tangent line is horizontal at $x = e$, and there are no other critical numbers for the function, the function *must* lie below its tangent line near $x = e$. Read that again and see why are the conditions listed are necessary. You might want to draw the function above the tangent line at $x = e$ and see how that leads to contradictions. The function must therefore look something like:



Therefore, the function has an absolute max of $1/e$ at $x = e$.

Example If a and b are positive numbers, find the maximum value of $f(x) = x^a(1-x)^b,\ 0 \le x \le 1$.

The derivative will always exist since a and b are positive (if they could be negative, we could have a denominator other than 1).

The only critical numbers will be if $f'(c) = 0$:

$$\begin{aligned}
f(x) &= x^a(1-x)^b \\
f'(x) &= \frac{d}{dx}\left[x^a(1-x)^b\right] \\
&= x^a\frac{d}{dx}\left[(1-x)^b\right] + (1-x)^b\frac{d}{dx}\left[x^a\right] \\
&= bx^a(1-x)^{b-1}(-1) + a(1-x)^b x^{a-1} \\
&= a(1-x)^b x^{a-1} - bx^a(1-x)^{b-1} \\
&= a(1-x)^b x^a x^{-1} - bx^a(1-x)^b(1-x)^{-1} \\
&= (1-x)^b x^a\left(ax^{-1} - b(1-x)^{-1}\right) \\
&= (1-x)^b x^a\left(\frac{a}{x} - \frac{b}{1-x}\right) \\
&= (1-x)^b x^a\left(\frac{a(1-x)-bx}{x(1-x)}\right) \\
&= (1-x)^{b-1}x^{a-1}\left(a-(a+b)x\right)
\end{aligned}$$

The critical number is therefore $c = \dfrac{a}{a+b}$.

Now, we evaluate the function at the critical numbers in the interval and at the endpoints of the interval:

$$\begin{aligned}
f\left(\frac{a}{a+b}\right) &= \left(\frac{a}{a+b}\right)^a\left(1-\left(\frac{a}{a+b}\right)\right)^b \\
&= \left(\frac{a}{a+b}\right)^a\left(\frac{b}{a+b}\right)^b > 0 \text{ (since } a \text{ and } b \text{ are positive)} \\
f(0) &= 0^a(1-0)^b = 0 \\
f(1) &= 1^a(1-1)^b = 0
\end{aligned}$$

Therefore, the maximum value of $f(x) = x^a(1-x)^b,\ 0 \le x \le 1$ is $\left(\dfrac{a}{a+b}\right)^a\left(\dfrac{b}{a+b}\right)^b$ which occurs at $x = \dfrac{a}{a+b}$.