Questions

Example Find the local maximum and minimum values of $f(x) = x^5 - 5x + 3$ using both the First and Second Derivative Tests. Which method do you prefer?

Example For $f(x) = \ln(1 - \ln x)$,
(a) find any vertical and horizontal asymptotes
(b) find the intervals of increase or decrease
(c) find any local maximum or minimum values
(d) find the intervals of concavity and any inflection points
(e) sketch the graph of $f(x)$

Solutions

Example Find the local maximum and minimum values of $f(x) = x^5 - 5x + 3$ using both the First and Second Derivative Tests. Which method do you prefer?

First derivative test:

• Intervals of Increasing/Decreasing:
Solve $f'(c) = 5c^4 - 5 = 0$. The real valued solutions to this equation are $c = -1, +1$. These are the only critical numbers for $f'(x)$ since $f'(x)$ exists for all x.

Write down a table showing where $f(x)$ is increasing and decreasing:

Interval	$f'(a)$ (a is in interval)	Sign of f'	f
$(-\infty, -1)$	$f'(-2) = 5(-2)^4 - 5 = 75$	$+$	increasing
$(-1, 1)$	$f'(0) = 5(0)^4 - 5 = -5$	$-$	decreasing
$(1, \infty)$	$f'(2) = 5(2)^4 - 5 = 75$	$+$	increasing

• Max/Min:
f goes from increasing to decreasing at $x = -1 \longrightarrow$ local max.
$f(-1) = (-1)^5 - 5(-1) + 3 = 7$. Point: $(-1, 7)$
f goes from decreasing to increasing at $x = +1 \longrightarrow$ local min.
$f(+1) = (+1)^5 - 5(+1) + 3 = -1$. Point: $(+1, -1)$

Second derivative test:

Solve $f'(c) = 5c^4 - 5 = 0$. The real valued solutions to this equation are $c = -1, +1$. These are the only critical numbers for $f'(x)$ since $f'(x)$ exists for all x.

We then evaluate the second derivative at the critical numbers to determine if we have a max or min.
$f''(x) = 20x^3$
$f''(-1) = 20(-1)^3 < 0 \longrightarrow f(x)$ is concave down at $x = -1$, therefore we have a local max at $x = -1$. $f(-1) = (-1)^5 - 5(-1) + 3 = 7$. Point: $(-1, 7)$
$f''(+1) = 20(+1)^3 > 0 \longrightarrow f(x)$ is concave up at $x = +1$, therefore we have a local min at $x = +1$. $f(+1) = (+1)^5 - 5(+1) + 3 = -1$. Point: $(+1, -1)$

I prefer the Second Derivative Test, since it is quicker if it works. However, I also like the First Derivative Test since it reminds us about the importance of the first derivative as to increasing or decreasing.

Example For $f(x) = \ln(1 - \ln x)$,
(a) find any vertical and horizontal asymptotes
(b) find the intervals of increase or decrease
(c) find any local maximum or minimum values
(d) find the intervals of concavity and any inflection points
(e) sketch the graph of $f(x)$

The goal here is to sketch the function using calculus, without the aid of a computer. We will need the derivatives, so let's get them first:

$$\begin{aligned}
f(x) &= \ln(1 - \ln x) \\
f'(x) &= \frac{d}{dx}[\ln(1 - \ln x)] \\
&= \frac{1}{1 - \ln x}\frac{d}{dx}[1 - \ln x] \text{ (chain rule)} \\
&= \frac{1}{1 - \ln x}\left(-\frac{1}{x}\right) \\
&= -\frac{1}{x(1 - \ln x)} \\
f''(x) &= -\frac{d}{dx}\left[\frac{1}{x(1 - \ln x)}\right] \\
&= -\frac{x(1 - \ln x)\frac{d}{dx}[1] - 1\frac{d}{dx}[x(1 - \ln x)]}{x^2(1 - \ln x)^2} \\
&= -\frac{x(1 - \ln x)(0) - (1 - \ln x)\frac{d}{dx}[x] - x\frac{d}{dx}[(1 - \ln x)]}{x^2(1 - \ln x)^2} \\
&= -\frac{-(1 - \ln x) - x\left(-\frac{1}{x}\right)}{x^2(1 - \ln x)^2} \\
&= -\frac{-1 + \ln x + 1}{x^2(1 - \ln x)^2} \\
&= -\frac{\ln x}{x^2(1 - \ln x)^2}
\end{aligned}$$

• Horizontal Asymptotes:

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} (\ln(1 - \ln x)) \longrightarrow \ln(-\infty)$$

To get the horizontal asymptotes, we need to know what happens to our function as $x \to \infty$ and $x \to -\infty$. We tried to do that above, and ran into a problem, since $\ln(-\infty)$ is not defined. This clues us in that maybe we should look at the domain of our function before proceeding.

Since $\ln x$ is only defined for $x > 0$, we know our function must have $x > 0$ due to the red part in $f(x) = \ln(1 - \ln x)$. Also, because of the blue part of $f(x) = \ln(1 - \ln x)$, we must have that $1 - \ln x > 0$. This means

$$\begin{aligned}
1 - \ln x &> 0 \\
\ln x &< 1 \\
x &< e^1 = e
\end{aligned}$$

So the domain of our function is $0 < x < e$, and there are no horizontal asymptotes since the function is not defined outside this region.

• Vertical Asymptotes:

$$\lim_{x\to a} f(x) = \pm\infty \rightsquigarrow x = a \text{ is a vertical asymptote}$$

Our function $f(x)$ is continuous, so the only place we might have a vertical asymptote is is at the endpoints. Let's check them:

$$\begin{aligned} \lim_{x\to 0} f(x) &= \lim_{x\to 0} \ln(1-\ln x) \to \ln(1-(-\infty)) \to +\infty \\ \lim_{x\to e} f(x) &= \lim_{x\to e} \ln(1-\ln x) \to \ln(1-1) \to \ln 0 \to -\infty \end{aligned}$$

We have vertical asymptotes at both endpoints, $x = 0$ and $x = e$.

• Intervals of Increasing/Decreasing:
Solve $f'(c) = -\frac{1}{c(1-\ln c)} = 0$. This condition does not occur inside our interval. Also, $f'(x)$ exists for all x. There are no critical numbers for $f'(x)$.

Write down a table showing where $f(x)$ is increasing and decreasing:

Interval	$f'(a)$ (a is in interval)	Sign of f'	f
$(0, e)$	$f'(1) = -\frac{1}{(1)(1-\ln 1)} = -1$	$-$	decreasing

• Max/Min:
Since the function is always decreasing on $(0, e)$, there are no max or mins.

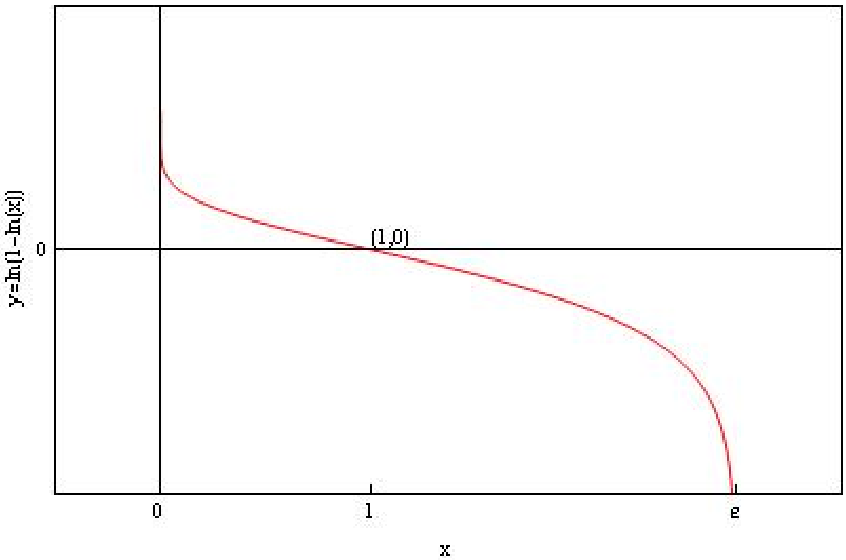
• Intervals of Concave Up/Concave Down:
Solve $f''(c) = -\frac{\ln c}{c^2(1-\ln c)^2} = 0$. The only solution is $c = +1$, since the numerator is zero there and the denominator is finite. This is the only critical number for $f''(x)$ since $f''(x)$ exists for all x.

Write down a table showing where $f(x)$ is concave up and down. We will need to use the fact that $\ln x < 0$ if $x < 1$, and $\ln x > 0$ is $x > 1$ to help us get the sign of f'' is the intervals.

Interval	$f''(a)$ (a is in interval)	Sign of f''	f
$(0, 1)$	$f''(1/2) = -\frac{\ln 1/2}{(1/2)^2(1-\ln 1/2)^2} = -\frac{-}{+} > 0$	$+$	Concave Up
$(1, e)$	$f''(3/2) = -\frac{\ln 3/2}{(3/2)^2(1-\ln 3/2)^2} = -\frac{+}{+} < 0$	$-$	Concave Down

• Points of Inflection:
The function f goes from concave up to concave down at $x = 1 \longrightarrow$ point of inflection. $f(1) = \ln(1-\ln 1) = \ln(1-0) = 0$. Point: $(1, f(1)) = (1, 0)$ (Hey! This means $x = 1$ is a root of f!)

• Sketch: Putting everything together from our detailed analysis, we get


y=ln(1-ln(x))
0
(1,0)
0
1
e
x