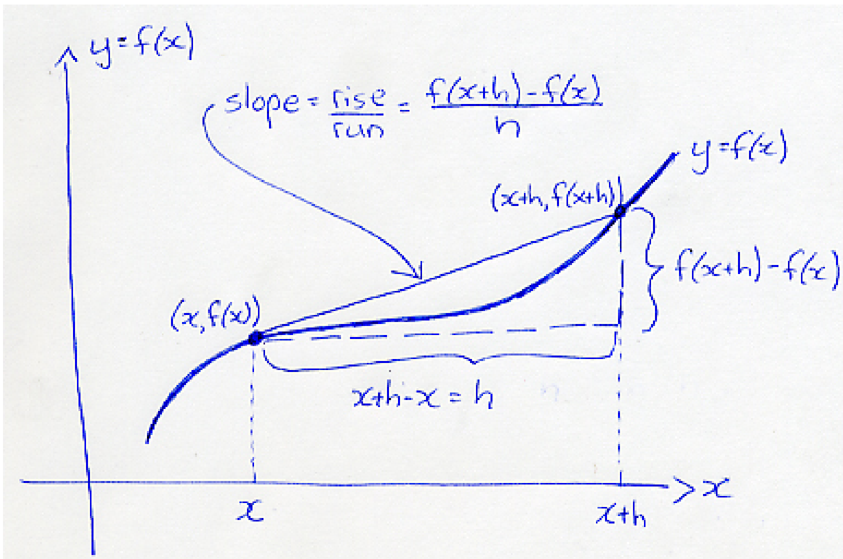
Average and Instantaneous Rate of Change

The *average rate of change of a function f over the interval* $(x, x+h)$ is given by

$$\text{Average Rate of Change} = \frac{f(x+h) - f(x)}{h}. \tag{1}$$



In differential calculus, you will be interested in calculating the *instantaneous rate of change of f at the point x* (from the sketch above, you might be able to guess this would be equal to the slope of the tangent line to f at x). This means you will be calculating expressions like

$$\text{Instantaneous Rate of Change} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}. \tag{2}$$

Notice that if we try to evaluate this limit by simply substituting $h = 0$, we get $\frac{0}{0}$, which is an indeterminant form (we will look at indeterminant forms in more detail in Section 2.5). This is why calculus begins with the study of limits, where you learn how to treat limits that lead to indeterminant forms.

In calculus, for certain functions f you will work out the instantaneous rate of change using Eq. (2) and algebra (for other functions f, you will develop more advanced techniques that allow you to work out the instantaneous rate of change without using Eq. (2)).

The following sees what happens to the average rate of change for the 12 basic functions when we try to simplify Eq (1). This motivates why we need to have a complete understanding of the 12 basic functions, superb algebra skills, and an excellent understanding of functional notation to do well in calculus. These are the skills precalculus can help you build. *Note that some of the simplifications that follow involve algebraic techniques we will be learning as the course proceeds–I've labeled these functions* **foreshadowing**.

Notice that there are three algebra techniques required so substituting $h = 0$ does not give $\frac{0}{0}$: factoring (for powers), getting a common denominator (for fractions), and rationalizing (for square roots). Although you can compute an average rate of change without all the simplifying, it is good practice to simplify whenever you can. Remember, a significant goal in precalculus is to build your skills in working with functional notation, and algebra.

Identity Function $f(x) = x$

$$\begin{aligned}
\text{Average Rate of Change} &= \frac{f(x+h)-f(x)}{h} \\
&= \frac{(x+h)-(x)}{h} \\
&= \frac{h}{h} \\
&= 1 \\
\text{Instantaneous Rate of Change} &= \lim_{h\to 0} 1 = 1
\end{aligned}$$

Squaring Function $f(x) = x^2$

$$\begin{aligned}
\text{Average Rate of Change} &= \frac{f(x+h)-f(x)}{h} \\
&= \frac{(x+h)^2-(x)^2}{h} \qquad \text{algebra technique: factor out} \\
&= \frac{x^2+2xh+h^2-x^2}{h} \\
&= \frac{2xh+h^2}{h} \\
&= \frac{h(2x+h)}{h} \\
&= 2x+h \\
\text{Instantaneous Rate of Change} &= \lim_{h\to 0}(2x+h) = 2x
\end{aligned}$$

Cubing Function $f(x) = x^3$

$$\begin{aligned}
\text{Average Rate of Change} &= \frac{f(x+h)-f(x)}{h} \\
&= \frac{(x+h)^3-(x)^3}{h} \\
&= \frac{x^3+3x^2h+3xh^2+h^3-x^3}{h} \qquad \text{algebra technique: factor out} \\
&= \frac{3x^2h+3xh^2+h^3}{h} \\
&= \frac{h(3x^2+3xh+h^2)}{h} \\
&= 3x^2+3xh+h^2 \\
\text{Instantaneous Rate of Change} &= \lim_{h\to 0}(3x^2+3xh+h^2) = 3x^2
\end{aligned}$$

Reciprocal Function $f(x) = \dfrac{1}{x}$

$$
\begin{aligned}
\text{Average Rate of Change} &= \frac{f(x+h) - f(x)}{h} \\
&= \frac{\left(\frac{1}{x+h}\right) - \left(\frac{1}{x}\right)}{h} \\
&= \frac{1}{h}\left(\frac{1}{x+h} - \frac{1}{x}\right) \qquad \text{algebra technique: common denominator} \\
&= \frac{1}{h}\left(\frac{x}{(x+h)x} - \frac{x+h}{(x+h)x}\right) \\
&= \frac{1}{h}\left(\frac{x-(x+h)}{(x+h)x}\right) \\
&= \frac{1}{h}\left(\frac{x-x-h}{(x+h)x}\right) \\
&= \frac{1}{h}\left(\frac{-h}{(x+h)x}\right) \\
&= \frac{-1}{(x+h)x} \\
\text{Instantaneous Rate of Change} &= \lim_{h\to 0}\left(\frac{-1}{(x+h)x}\right) = -\frac{1}{x^2}
\end{aligned}
$$

Square Root Function $f(x) = \sqrt{x}$

$$
\begin{aligned}
\text{Average Rate of Change} &= \frac{f(x+h) - f(x)}{h} \\
&= \frac{\sqrt{x+h} - \sqrt{x}}{h} \qquad \text{algebra technique: rationalize numerator} \\
&= \frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}} \\
&= \frac{(\sqrt{x+h} - \sqrt{x})(\sqrt{x+h} + \sqrt{x})}{h(\sqrt{x+h} + \sqrt{x})} \\
&= \frac{x+h-x}{h(\sqrt{x+h} + \sqrt{x})} \\
&= \frac{h}{h(\sqrt{x+h} + \sqrt{x})} \\
&= \frac{1}{\sqrt{x+h} + \sqrt{x}} \\
\text{Instantaneous Rate of Change} &= \lim_{h\to 0}\left(\frac{1}{\sqrt{x+h} + \sqrt{x}}\right) = \frac{1}{2\sqrt{x}}
\end{aligned}
$$

Exponential Function $f(x) = e^x$ **(foreshadowing)**

$$\begin{aligned}
\text{Average Rate of Change} &= \frac{f(x+h) - f(x)}{h} \\
&= \frac{e^{x+h} - e^x}{h} \\
&= \frac{e^x e^h - e^x}{h} \\
&= e^x \left(\frac{e^h - 1}{h} \right)
\end{aligned}$$

The instantaneous rate of change requires techniques from calculus. Essentially, we define e to be the number such that $\lim_{h \to 0} \left(\frac{e^h - 1}{h} \right) = 1$.

Natural Logarithm Function $f(x) = \ln x$ **(foreshadowing)**

$$\begin{aligned}
\text{Average Rate of Change} &= \frac{f(x+h) - f(x)}{h} \\
&= \frac{\ln(x+h) - \ln x}{h} \text{ (can not be simplified any further)}
\end{aligned}$$

The instantaneous rate of change requires techniques from calculus. It is not calculated from this definition.

Sine Function $f(x) = \sin x$ **(foreshadowing)**

$$\begin{aligned}
\text{Average Rate of Change} &= \frac{f(x+h) - f(x)}{h} \\
&= \frac{\sin(x+h) - \sin x}{h} \\
&= \frac{\sin x \cos h + \cos x \sin h - \sin x}{h} \\
&= \frac{\sin x(\cos h - 1) + \cos x \sin h}{h} \\
&= \sin x \left(\frac{\cos h - 1}{h} \right) + \cos x \left(\frac{\sin h}{h} \right)
\end{aligned}$$

The instantaneous rate of change requires us to evaluate the two limits $\lim_{h \to 0} \left(\frac{\cos h - 1}{h} \right)$ and $\lim_{h \to 0} \left(\frac{\sin h}{h} \right)$ which is done in calculus.

Cosine Function $f(x) = \cos x$ **(foreshadowing)**

$$\begin{aligned}
\text{Average Rate of Change} &= \frac{f(x+h)-f(x)}{h} \\
&= \frac{\cos(x+h)-\cos x}{h} \\
&= \frac{\cos x \cos h - \sin x \sin h - \cos x}{h} \\
&= \frac{\cos x(\cos h - 1) - \sin x \sin h}{h} \\
&= \cos x\left(\frac{\cos h - 1}{h}\right) - \sin x\left(\frac{\sin h}{h}\right)
\end{aligned}$$

The instantaneous rate of change requires us to evaluate the two limits $\lim_{h\to 0}\left(\frac{\cos h - 1}{h}\right)$ and $\lim_{h\to 0}\left(\frac{\sin h}{h}\right)$ which is done in calculus.

Absolute Value Function $f(x) = |x|$ **(foreshadowing)**

$$\begin{aligned}
\text{Average Rate of Change} &= \frac{f(x+h)-f(x)}{h} \\
&= \frac{|x+h|-|x|}{h} \\
&= \begin{cases} \frac{x+h-x}{h} & \text{if } x+h \geq 0 \\ \frac{-(x+h)-x}{h} & \text{if } x+h < 0 \end{cases} \\
&= \begin{cases} 1 & \text{if } x+h \geq 0 \\ -1 & \text{if } x+h < 0 \end{cases}
\end{aligned}$$

The instantaneous rate of change requires some subtle concepts from the ideas of limits which are studied in calculus.

Greatest Integer Function $f(x) = \mathbf{int}(x)$

The greatest integer function requires some special concepts from the study of limits to treat the instantaneous rate of change properly. It is best left to a calculus class to look at the instantaneous rate of change for this function.

Logistic Function $f(x) = \dfrac{1}{1+e^{-x}}$

The instantaneous rate of change for the logistic function is best calculated using the special techniques studied in calculus. The instantaneous rate of change is not calculated from Eq. (2) for the logistic function.