- This handout is not meant to be comprehensive. Study concept check, true/false quiz, and review exercises from text.
- Know the basic concepts from Chapter 1 (logarithms, exponentials, functional notation, algebra, etc).

Main concepts:

- limits and limit laws (evaluating limits, techniques when $\to 0/0$ or $\infty - \infty$, indeterminant forms)
- intermediate value theorem (roots of equations)
- continuity, discontinuity (jump, removable, infinite)
- vertical and horizontal asymptotes (infinite limits, limits at infinity)
- tangent lines, velocities, and other rates of change
- derivative (two definitions for derivative at a point, $g'(2)$)
- derivative as a function (given $f(x)$, find $f'(x)$, both algebraically and graphically)

Be comfortable working with:

$$\lim_{x\to a} f(x) = L$$

$$\lim_{x\to a^+} f(x) = L$$

$$\lim_{x\to a^-} f(x) = L$$

$$\lim_{x\to a} f(x) = L \longleftrightarrow \lim_{x\to a^-} f(x) = L \text{ and } \lim_{x\to a^+} f(x) = L$$

$$\lim_{x\to a} f(x) = \infty \text{ (the limit does not exist)}$$

- vertical asymptote: $x = a$ is a vertical asymptote if $\lim_{x\to a^{+/-}} f(x) = \pm\infty$
- horizontal asymptote: $y = L$ is a horizontal asymptote if either $\lim_{x\to\pm\infty} f(x) = L$
- Indeterminant forms mean there is more work to do: indeterminant quotient: $\frac{0}{0}$, $\frac{\infty}{\infty}$. Indeterminant difference: $\infty - \infty$.
- Limit techniques: factor, rationalize, common denominator:

$$\lim_{h\to 0} \frac{(3+h)^2 - 9}{h} = \lim_{h\to 0} \frac{9 + 6h + h^2 - 9}{h}$$

$$\lim_{t\to 0} \frac{\sqrt{t^2+9} - 3}{t^2} = \lim_{t\to 0} \frac{\sqrt{t^2+9} - 3}{t^2} \cdot \frac{\sqrt{t^2+9} + 3}{\sqrt{t^2+9} + 3}$$

$$\lim_{x\to 2} \frac{\left(\frac{1}{x} - \frac{1}{2}\right)}{x-2} = \lim_{x\to 2} \frac{1}{x-2}\left[\frac{2-x}{2x}\right]$$

- If $r > 0$ is a rational number, then $\displaystyle\lim_{x\to\infty} \frac{1}{x^r} = 0$.
- If $r > 0$ is a rational number such that x^r is defined for all x, then $\displaystyle\lim_{x\to-\infty} \frac{1}{x^r} = 0$.

$$\lim_{x\to\infty} \frac{3x^2 - x - 2}{5x^2 + 4x + 1} = \lim_{x\to\infty} \frac{3 - \frac{1}{x} - \frac{2}{x^2}}{5 + \frac{4}{x} + \frac{1}{x^2}}$$

Techqnique: To evaluate a limit at infinity of a rational function, we divide the numerator and denominator by the highest power of x that occurs in the denominator.

$$\lim_{x\to-\infty} \sqrt{x^2 - x} + x = \lim_{x\to-\infty} \sqrt{x^2 - x} + x \cdot \frac{\sqrt{x^2 - x} - x}{\sqrt{x^2 - x} - x} \text{ (rationalize, then divide by highest power of } x \text{ in denominator)}$$

If the limit is as $x \to -\infty$ and involves square roots, remember $x = -\sqrt{x^2}$ if $x < 0$.

• Continuity: The function f is continuous at a if $\lim_{x \to a} f(x) = f(a)$.

• Continuous from the left and continuous from the right.

• Polynomials are continuous everywhere, ie. $P(x)$ is a polynomial then it is continuous for $x \in (-\infty, \infty)$.

• Continuous on their domains: polynomials, rational, root, trig, inverse trig, exponential, and logarithmic functions.

Example: infinite limits at infinity Find $\lim_{x \to \infty} (x^2 - x)$

$$\begin{aligned} \lim_{x \to \infty} (x^2 - x) &\to \infty - \infty \\ &= \lim_{x \to \infty} x(x-1) = \infty \cdot \infty = \infty \end{aligned}$$

Definition The tangent line to $y = f(x)$ at $(a, f(a))$ is the line through $(a, f(a))$ whose slope is equal to $f'(a)$, the derivative of f evaluated at a.

• The point-slope form of the equation of a line with slope m through the point (x_1, y_1): $y - y_1 = m(x - x_1)$

• The derivative of $f(x)$ at $x = a$ can be interpreted as the slope of the tangent line to the curve at $x = a$ which is the slope of the curve at $x = a$.

• The derivative of the position function $s = f(t)$ is the velocity function $f'(t) = v(t)$.

• The derivative is an instantaneous rate of change.

Example A particle moves along a straight line with equation of motion $s = f(t) = 2t^3 - t$, where s is measured in meters and t in seconds. Find the velocity when $t = 2$.

• The derivative as function:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

• A function can be continuous at a point, but not differentiable at that point.

• A function which is differentiable at a point is continuous at that point.

Example draw a sketch which illustrates why the definition

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

can be interpreted as the slope of the tangent line at $x = a$.

Theorems

Theorem If f and g are continuous at a and c is a constant, then $f + g$, $f - g$, cf, fg, f/g if $g(a) \neq 0$ are also continuous at a.

Theorem If f is continuous at b and $\lim_{x \to a} g(x) = b$, then $\lim_{x \to a} f(g(x)) = f\left(\lim_{x \to a} g(x)\right)$.

Theorem If g is continuous at a and f is continuous at $g(a)$, then $f \circ g$ is continuous at a.

Intermediate Value Theorem Suppose that f is continuous on the closed interval $[a, b]$ and let N be any number between $f(a)$ and $f(b)$. Then there exists a number c in (a, b) such that $f(c) = N$.

Example Show that there is a root of the equation $4x^3 - 6x^2 + 3x - 2 = 0$ between 1 and 2. Sketch the situation.