Math 1101 Calculus I

~~Quiz 5: Tue March 22 2011~~ **Name(s)** Solution.

~~**Instructions:** You may work in pairs, and use your notes (but not *Mathematica*). Each pair turns in one solution, with both names. **No Regrade for this Quiz.**~~

(20 marks) Sketch $f(x) = e^{1/x}$ by showing a clear discussion of the following properties of $f(x)$ (note that $f(x)$ may not possess all of these things):

1. horizontal asymptotes
2. vertical asymptotes
3. zeroes
4. intervals of increasing/decreasing
5. extrema
6. intervals of concave up/concave down
7. points of inflection
8. a well labeled sketch

1. Horizontal Asymptotes

$$\lim_{x \to \infty} e^{1/x} = e^{0} = 1$$ So there is a horizontal asymptote of $y=1$.

$$\lim_{x \to -\infty} e^{1/x} = e^{0} = 1$$

2. Vertical Asymptotes

We have a division by zero at $x=0$, so we should check around $x=0$ carefully.

$$\lim_{x \to 0^+} e^{1/x} = e^{\infty} = \infty.$$ vertical asymptote at $x=0$ from right only.

$$\lim_{x \to 0^-} e^{1/x} = e^{-\infty} = 0$$

3. Zeros

Solve $e^{1/x} = 0$

$$\ln e^{1/x} = \ln 0$$

Since $\ln 0$ does not not exist, we have no zeros.

4. Intervals of inc/dec & 5. extrema

$$f'(x) = \frac{d}{dx}\left[e^{1/x}\right]$$
$$= \frac{d}{du}\left[e^{u}\right] \cdot \frac{du}{dx}, \quad u = 1/x$$
$$= e^{u}\left(-\frac{1}{x^2}\right)$$
$$= -\frac{1}{x^2} e^{1/x}$$

$f'(x) = 0$ has no solution

$f'(x)$ does not exist at $x=0$.

	$(-\infty, 0)$	0	$(0, \infty)$	
	$-$		$-$	$f'(x)$
		0		x
	dec		dec	$f(x)$

$f(x)$ is dec on $(-\infty, 0)$ and $(0, \infty)$. No extrema.

6. Intervals of concave up/down & 7. Points of Inflection

$$f''(x) = \frac{d}{dx}\left[-\frac{1}{x^2}e^{1/x}\right]$$

$$= -\frac{1}{x^2}\frac{d}{dx}\left[e^{1/x}\right] - e^{1/x}\frac{d}{dx}\left[\frac{1}{x^2}\right]$$

$$= -\frac{1}{x^2}\left(-\frac{1}{x^2}e^{1/x}\right) - e^{1/x}\left(\frac{-2}{x^3}\right)$$

$$= e^{1/x}\left(\frac{1}{x^4} + \frac{2}{x^3}\right)$$

$$= e^{1/x}\left(\frac{1+2x}{x^4}\right)$$

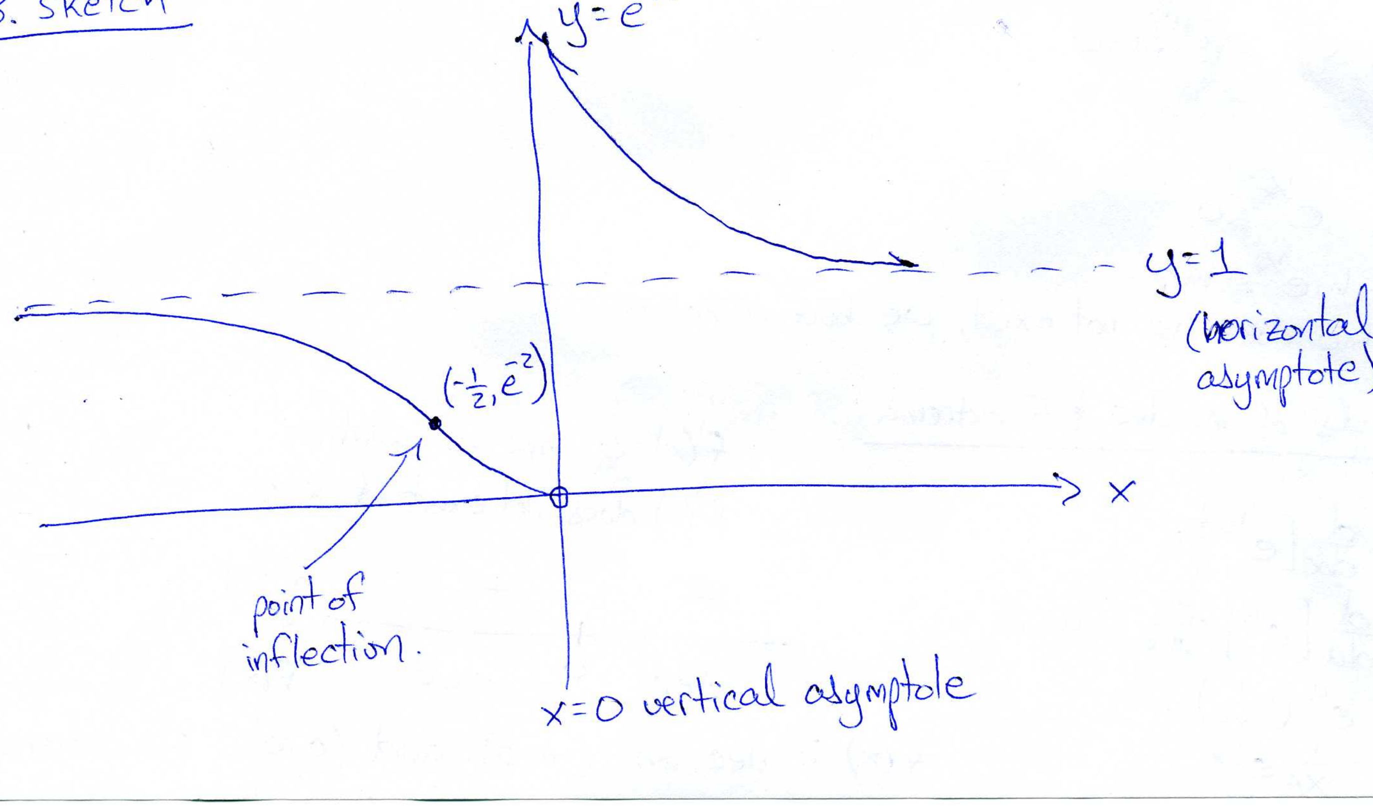
$f''(x) = 0$ when $1+2x = 0 \Rightarrow x = -1/2$.

$f''(x)$ does not exist when $x = 0$.

	$x < -\frac{1}{2}$	$-\frac{1}{2} < x < 0$	$x > 0$
$f''(x)$	$-$	$+$	$+$
$f(x)$	$\cap$	$\cup$	$\cup$

Point of inflection at $x = -1/2$, $y = f(-1/2) = e^{-2} \Rightarrow \left(-\frac{1}{2}, e^{-2}\right)$

8. sketch



Ex $f(x) = \dfrac{e^x}{x}$.

Note: no nice way to sketch using techniques from precalculus.

$$f'(x) = \frac{xe^x - e^x}{x^2} = \frac{e^x(x-1)}{x^2}$$

$$f''(x) = \frac{x^2\left[e^x(x-1) + (1)e^x\right] - e^x(x-1)2x}{x^4}$$

$$= \frac{x^2(e^x x - \cancel{e^x} + \cancel{e^x}) - 2e^x x^2 + 2e^x x}{x^4}$$

$$= \frac{e^x x^3 - 2e^x x^2 + 2e^x x}{x^4}$$

$$= \frac{e^x}{x^4}(x^3 - 2x^2 + 2x)$$

$$= \frac{e^x(x^2 - 2x + 2)}{x^3}$$

Note $x^2 - 2x + 2$ does not factor.

$$\boxed{\begin{aligned} f(x) &= \frac{e^x}{x} \\ f'(x) &= \frac{e^x(x-1)}{x^2} \\ f''(x) &= \frac{e^x(x^2-2x+2)}{x^3} \end{aligned}}$$

Horizontal Asymptotes:

$$\lim_{x\to\infty} \frac{e^x}{x} \to \frac{\infty}{\infty} \quad \text{L'Hospital}$$

$$= \lim_{x\to\infty} \frac{e^x}{1} = \infty.$$

$$\lim_{x\to-\infty} \frac{e^x}{x} \to \frac{0}{-\infty} = 0.$$

Horizontal Asymptote of $y=0$.

Vertical Asymptotes:

$f(x)$ is infinite at $x=0$.

$$\lim_{x\to0^+} \frac{e^x}{x} \to \frac{1}{+0} = \infty$$

$$\lim_{x\to0^-} \frac{e^x}{x} \to \frac{1}{-0} = -\infty.$$

Zeros:

$\dfrac{e^x}{x} = 0 \qquad e^x = 0$ no solution. no zeros.

if you like (sometimes this is hard to do)

f: $-$ on $(-\infty, 0)$, $+$ on $(0, \infty)$ (x-axis, mark at 0)

f: neg, pos.

Inc/Dec/extrema: $f'(x) = \dfrac{e^x(x-1)}{x^2}$

$f'(x) = 0$ or $f'(x)$ does not exist.

$e^x = 0$ or $x - 1 = 0$

$x = 1$

no solution

$x = 0$

f' $-$ $-$ $+$

x: 0 1

f dec dec inc

vertical asymptote!

min at $x = 1$

Point $(1, f(1)) = (1, e)$

concavity/pt inflection: $f''(x) = \dfrac{e^x(x^2-2x+2)}{x^3}$

$f''(x) = 0$ or $f''(x)$ does not exist.

$e^x = 0$ or $x^2 - 2x + 2 = 0$

no solution no solution.

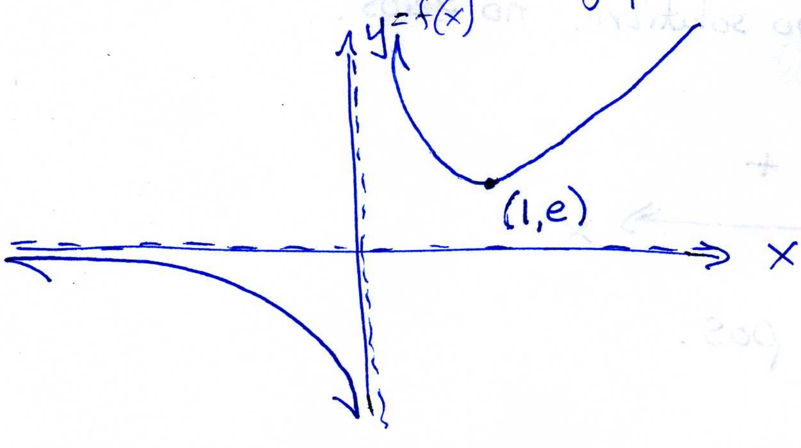
$x = 0$

f'' $-$ $+$

x: 0

f $\cap$ $\cup$

vertical asymptote, so no point of inflection.



$$\boxed{f(x) = \frac{x^2}{x^2-1}}$$

$$f'(x) = \frac{d}{dx}\left[\frac{x^2}{x^2-1}\right] = \frac{(x^2-1)(2x) - x^2(2x)}{(x^2-1)^2} = \frac{2x(-1)}{(x^2-1)^2}$$

$$\boxed{f'(x) = -\frac{2x}{(x^2-1)^2}}$$

$$f''(x) = \frac{d}{dx}\left[\frac{-2x}{(x^2-1)^2}\right] = \frac{(x^2-1)^2(-2) - (-2x)(2(x^2-1)2x)}{(x^2-1)^4}$$

$$= \frac{-2\cancel{(x^2-1)}\left((x^2-1) - 4x^2\right)}{(x^2-1)^{\cancel{4}\,3}}$$

$$= \frac{-2(-1-3x^2)}{(x^2-1)^3} = \boxed{\frac{2+6x^2}{(x^2-1)^3} = f''(x)}$$

Horizontal Asymptotes:

$$\lim_{x\to\infty} \frac{x^2}{x^2-1} \to \frac{\infty}{\infty}$$

$$= \lim_{x\to\infty} \frac{1}{1-1/x^2} = 1$$

Also $\lim_{x\to-\infty} \frac{x^2}{x^2-1} = \lim_{x\to-\infty} \frac{1}{1-1/x^2} = 1.$

So $y=1$ is a horizontal asymptote

Vertical Asymptotes:

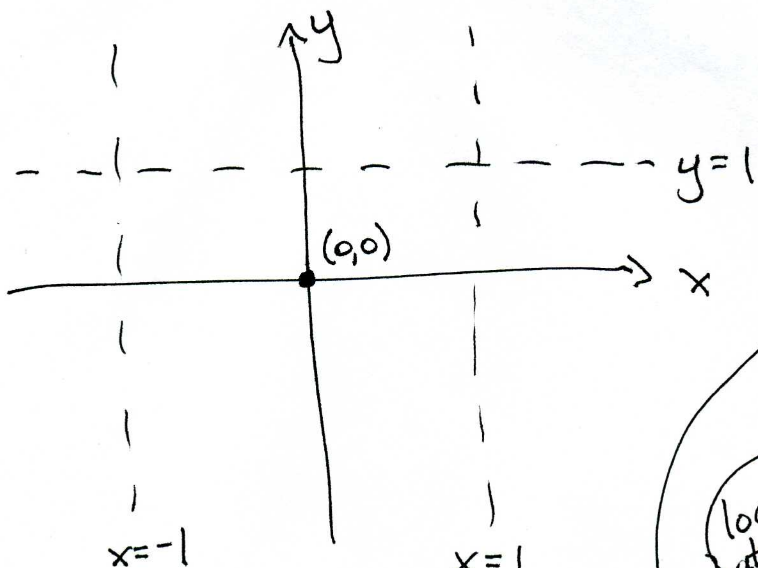
rational function, so look for Division by zero.

$$x^2-1=0 \Rightarrow x = \pm 1$$

Better to factor: $x^2-1 = (x+1)(x-1) = 0$

why? Multiplicity is odd, so f changes sign at vertical asymptotes.

Let's start a sketch:



Zeros: $\frac{x^2}{x^2-1} = 0 \Rightarrow x = 0.$

Point $(0,0)$.

Inc/Dec: $f'(x) = 0$ or does not exist.

$$\frac{-2x}{(x^2-1)^2} = 0 \Rightarrow x=0,\ x=-1,\ x=1$$

		-1		0		$+1$	
f'	$+$		$+$		$-$		$-$
f	inc		inc		dec		dec

local max at $x=0$. Point $(0,0)$.

Concavity: $f''(x)=0$ or $f''(x)$ does not exist.

$$\frac{2+6x^2}{(x^2-1)^3}=0$$

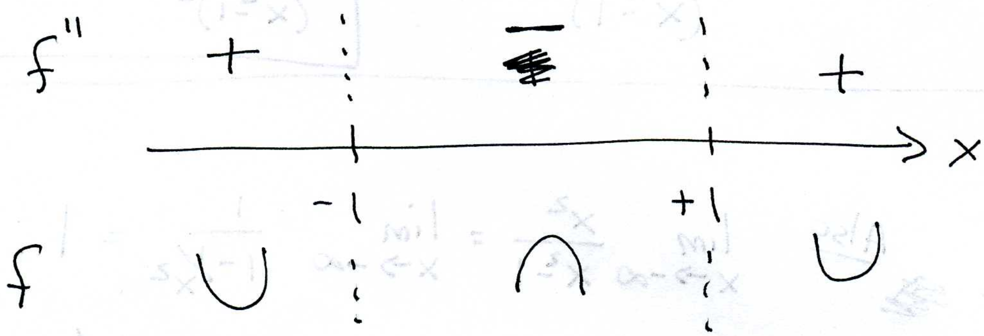
$2+6x^2=0$

$x^2=-\frac{1}{3}$

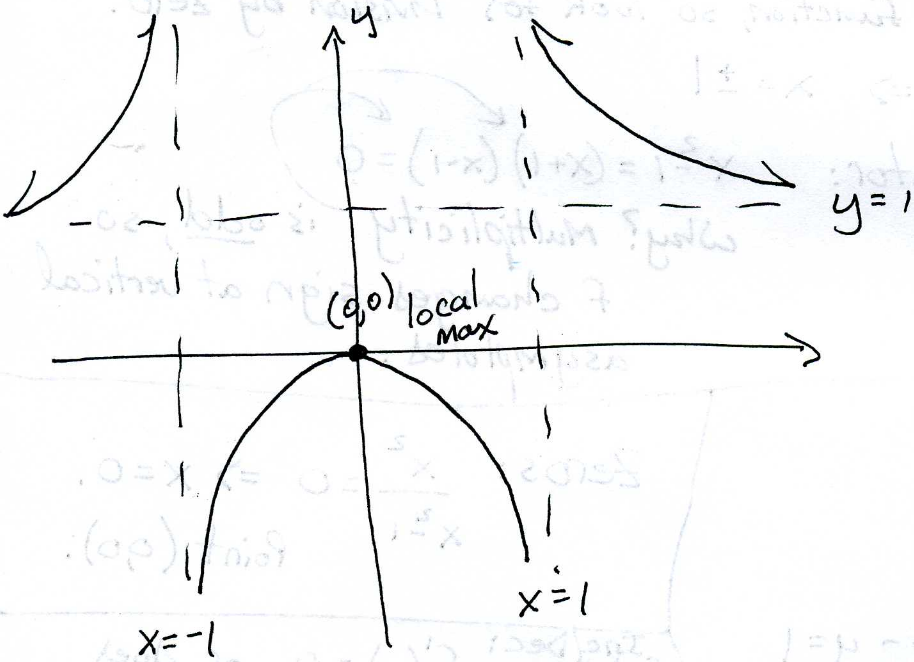
no solution.

$f''(x)$ does not exist

$x=\pm 1$.



Note: although concavity changes, there are no points of inflection. (concavity changes at the vertical asymptotes)



Draw other possibilities and have them explain why they aren't ~~possible~~ correct.

2(-1) - 3 + 12

16 - 12 - 24

Ex Sketch $f(x) = 2x^3 - 3x^2 - 12x$

Derivatives: $f'(x) = 6x^2 - 6x - 12$

$f''(x) = 12x - 6$

Horizontal Asymptotes

$\lim_{x \to \infty} f(x) = +\infty \qquad \lim_{x \to -\infty} f(x) = -\infty$. Why?

no horizontal asymptotes.

Vertical Asymptotes none, it's a polynomial!

Zeros $f(x) = 0 = 2x^3 - 3x^2 - 12x$

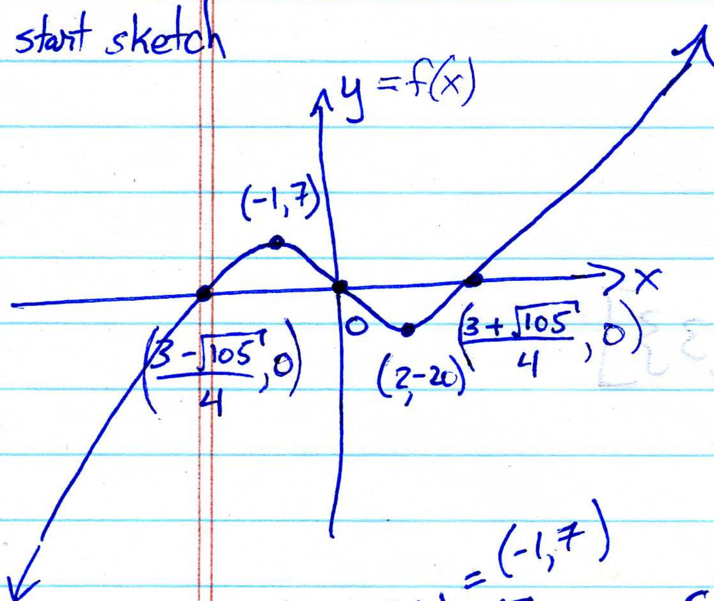
$0 = x(2x^2 - 3x - 12)$

$0 = x$ or $2x^2 - 3x - 12 = 0$

quadratic formula: $x = \dfrac{+3 \pm \sqrt{9 - 4(2)(-12)}}{2(2)}$

$= \dfrac{3 \pm \sqrt{105}}{4}$

start sketch



inc/dec/extrema $f'(x) = 0$ or ~~$f'(x)$ does not exist~~

$6x^2 - 6x - 12 = 0$

$x^2 - x - 2 = 0$

$x = \dfrac{1 \pm \sqrt{1 - 4(1)(-2)}}{2(1)}$

$= \dfrac{1 \pm 3}{2}$

$x = 2, -1$.

max $(-1, f(-1)) = (-1, 7)$

min $(2, f(2)) = (2, -20)$

f': + on $(-\infty, -1)$, − on $(-1, 2)$, + on $(2, \infty)$

f: inc ∩ dec ∪ inc

concavity on next page

concavity : $f''(x)=0$ $\quad$ ~~$f''(x)$ does not exist~~

$\hookrightarrow 12x-6=0$

$x = \frac{1}{2}$ pt. of inflection. $\left(\frac{1}{2}, f\left(\frac{1}{2}\right)\right) = \left(\frac{1}{2}, -\frac{13}{2}\right)$

f'' $\quad - \quad | \quad +$

f $\quad \cap \quad \frac{1}{2} \quad \cup$

$$2\left(\tfrac{1}{2}\right)^3 - 3\left(\tfrac{1}{2}\right)^2 - 12\left(\tfrac{1}{2}\right) = \frac{-13}{2}$$

MMA to check

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f[x_] = 2x^3 - 3x^2 - 12x

f'[x]
f''[x]

Solve[f[x] == 0, x]
    f'[x]
    f''[x]

f[-1]
f[2]
f[1/2]
Plot[f[x], {x, -3, 3}]
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