# 1101 Calculus I Section 1.1 Functions

A function $f$ is a rule that assigns to each element $x$ in a set $D$ exactly one element, called $f(x)$, in a set $R$.

The **range** $R$ is the set of all possible values of $f(x)$, when $x$ varies over the entire **domain** $D$.

The functions we consider have the domain and range as real numbers, denoted $\mathbb{R}$.

We denote this set as $x \in \mathbb{R}$, which is equivalent to $x \in (-\infty, \infty)$, which is equivalent to $-\infty < x < \infty$.

The symbol ($x$ in this case) which represents an arbitrary element in the domain of $f$ is called the *independent variable.*

The symbol ($f(x)$ in this case) which represents an arbitrary element in the range of $f$ is called the *dependent variable* (dependent because it depends on the value of $x$).

We often use $y = f(x)$ as dependent variable. This notation is called *Euler's function notation.* This is read as "$y$ equals $f$ of $x$". Note that this is not multiplication, that is, $f(x)$ does not mean $f$ times $x$.

We call $y = f(x)$ an *explicit function.*

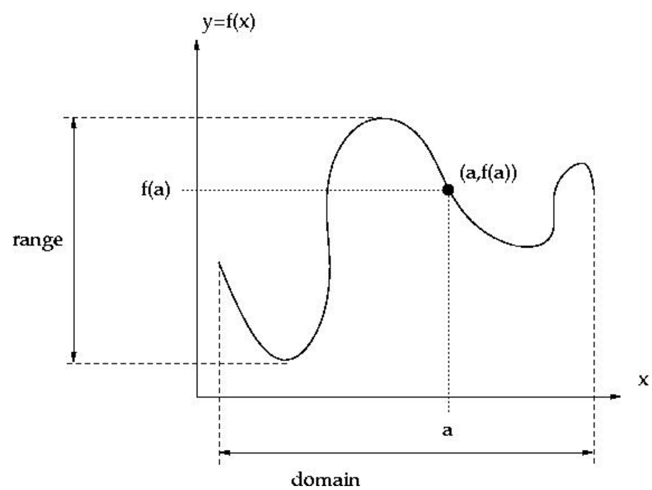
**Graph** A graph pictorially represents the relationship between ordered pairs, where the first element in the pair is from the domain, the second element from the range:

$$\{(x, f(x)) | x \in D\}$$

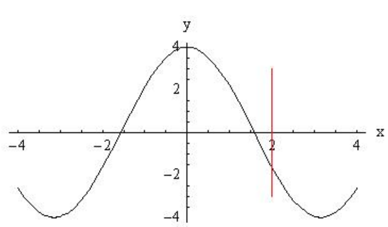
read: "ordered pair $(x, f(x))$ such that $x$ is an element of $D$ which is the domain."

The graph contains more information than the other descriptions.

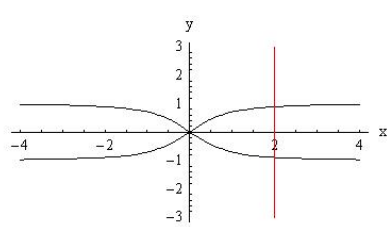
**General Example:**




**Vertical Line Test** A graph represents a function if every vertical line you can draw intersects the graph only once (this ensures we have exactly one element $f(x)$ for each $x$).

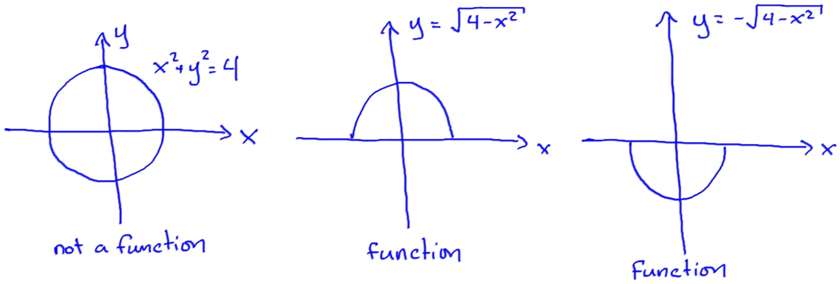



this graph represents a function




this graph does not represent a function

**Example** The relation $x^2 + y^2 = 4$ is not a function (it is what we call an *implicit function*), but it does consist of the two explicit functions.




**Example** Given $f(x) = x^2$, simplify the quantity $f(x+h) - f(x-h)$ as much as possible.

$$\begin{aligned} f(x+h) - f(x-h) &= (x+h)^2 - (x-h)^2 \\ &= (x^2 + h^2 + 2xh) - (x^2 + h^2 - 2xh) \\ &= x^2 + h^2 + 2xh - x^2 - h^2 + 2xh \\ &= 4xh \end{aligned}$$

**Example** Given $f(x) = \dfrac{x^2 - 1}{2x}$ determine $f(1)$, $f(2)$, $f(2+1)$.

$$\begin{aligned} f(1) &= \frac{(1)^2 - 1}{2(1)} = 0 \\ f(2) &= \frac{(2)^2 - 1}{2(2)} = \frac{3}{4} \\ f(2+1) &= \frac{(2+1)^2 - 1}{2(2+1)} = \frac{(3)^2 - 1}{2(3)} = \frac{4}{3} \end{aligned}$$

Notice that $f(2+1) = f(3)$, as you can simplify the quantity you are evaluating the function at.

However, $f(2+1) \neq f(2) + f(1)$.

This generalizes to the rule that $f(a+b) \neq f(a) + f(b)$, which leads to things like $\sqrt{a^2 + b^2} \neq a + b$ and $(a+b)^2 \neq a^2 + b^2$, among other things.

### More on Domain and Range

Given $y = f(x)$, the values of $x$ that can go into $f(x)$ and yield an output which is a real number form the domain. All the possible $y$'s that come out form the range.

**Example** Find the domain and range of $h(x) = \dfrac{\sqrt{4 - x^2}}{x - 5}$.

We cannot have division by zero, so we want to see where the denominator is zero and exclude that value of $x$ from the domain:

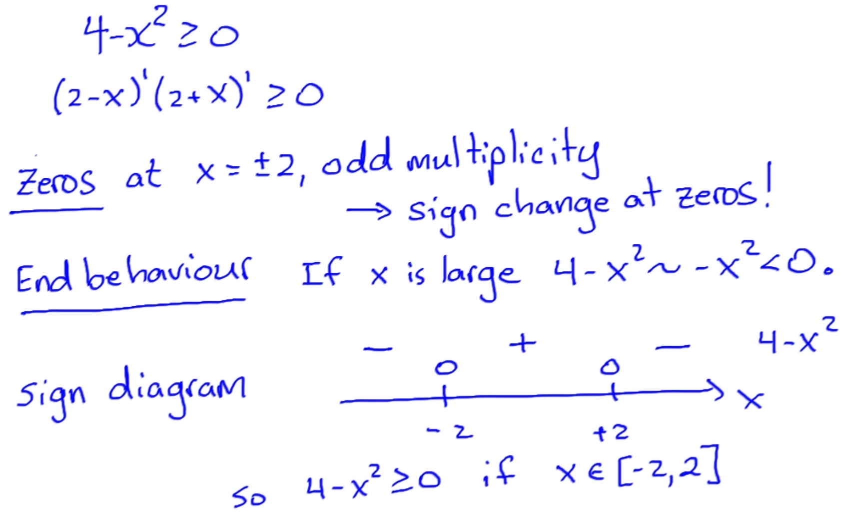
$$\begin{aligned} x - 5 &= 0 \\ x &= 5 \end{aligned}$$

so $x = 5$ is not in the domain.

We also cannot take the square root of a negative number and get a result which is a real number. So we must have values of $x$ for which

$$\begin{aligned} 4 - x^2 &\geq 0 \\ 4 &\geq x^2 \\ x^2 &\leq 4 \\ -\sqrt{4} \leq \sqrt{x^2} &\leq \sqrt{4} \\ -2 \leq x &\leq 2 \end{aligned}$$

You can also figure this out using a sign diagram, which in many cases is easier to implement than an algebraic solution to an inequality:




This means the domain of the function $h(x)$ is $-2 \leq x \leq 2$, or $x \in [-2, 2]$. The point $x = 5$ is excluded, but that is already contained in the restriction based on the square root.

The range is all possible output values. This is usually more complicated to figure out than the domain, but easy to find if we plot a graph. Using *Mathematica*, we find

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Plot[Sqrt[4-x^2]/(x-5),{x,-3,3}]
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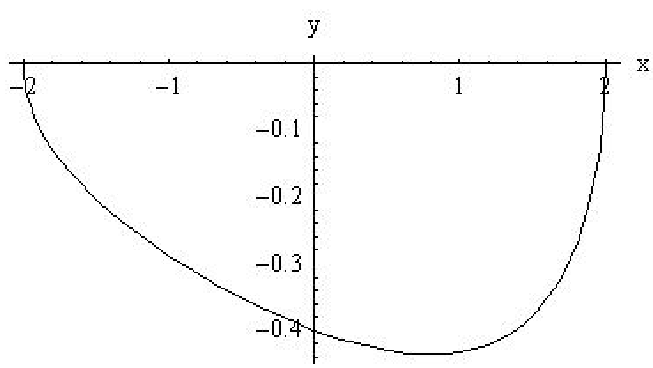
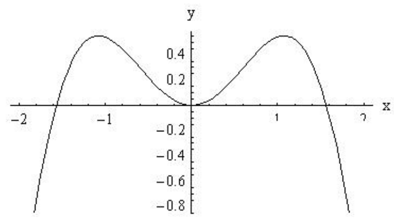


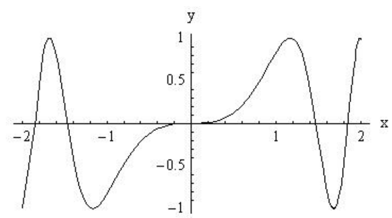

From the graph, we estimate the range to be $y \in [-0.44, 0]$. To get the range precisely we could use ideas from calculus, which we will learn later.

## Symmetry

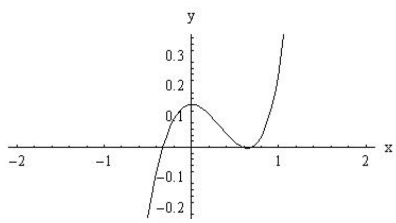
**Even** functions satisfy $f(-x) = f(x)$. Geometrically this means the function is symmetric about the $y$-axis.




**Odd** functions satisfy $f(-x) = -f(x)$. Geometrically this means the function is symmetric if we rotate 180 degrees about the origin.




NOTE: A function can be either even, or odd, or neither!



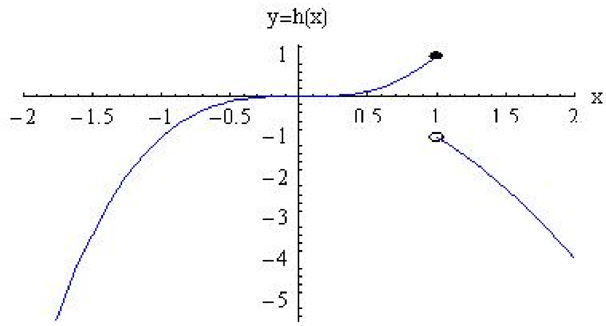

**Example** Determine whether the function $g(x) = \dfrac{x^3 - x}{x^3 + x}$ is even, odd, or neither. Use the algebraic technique to determine if a function is even or odd, rather than attempting to sketch the function.

$$\begin{aligned} g(-x) &= \frac{(-x)^3 - (-x)}{(-x)^3 + (-x)} \\ &= \frac{-x^3 + x}{-x^3 - x} \\ &= \frac{-(x^3 - x)}{-(x^3 + x)} \\ &= \frac{(x^3 - x)}{(x^3 + x)} = g(x) \end{aligned}$$

Since $g(-x) = g(x)$, the function $g$ is even.

**Piecewise Defined Functions** are defined by different formulas for different parts of their domains.

$$h(x) = \begin{cases} x^3 \text{ if } x \le 1 \\ -x^2 \text{ if } x > 1 \end{cases}$$




$$|x| = \begin{cases} x \text{ if } x \ge 0 \\ -x \text{ if } x < 0 \end{cases}$$