

1101 Calculus I Lecture 2.1: The Tangent and Velocity Problems

The Tangent Problem

A good way to think of what the tangent line to a curve is that it is *a straight line which approximates the curve well in the region where it touches the curve*. A more precise definition will be developed later.

Recall, straight lines have equations $y = mx + b$ (slope m , y -intercept b), or, more useful in this case, $y - y_0 = m(x - x_0)$ (slope m , and passes through the point (x_0, y_0)).

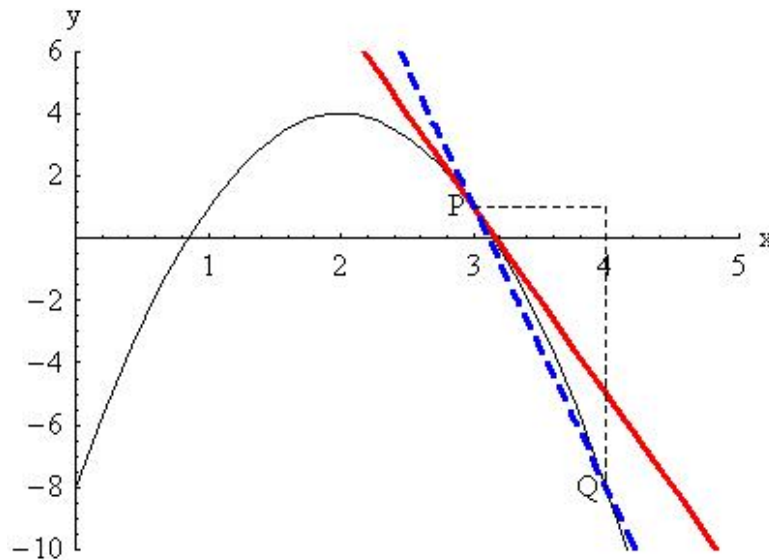
Your text has a fairly simple example. I will do something more complex instead.

Example Find the tangent line to the parabola $y = -3x^2 + 12x - 8$ at the point $P(3, 1)$.

Our solution involves finding the equation of a straight line, which is $y - y_0 = m(x - x_0)$. We already know the tangent line should touch the curve, so it will pass through the point $P(3, 1)$. This means $x_0 = 3$ and $y_0 = 1$. We now need to determine the slope of the tangent line, m .

But we need two points to determine the slope of a line, and we only know one. We only know that the tangent line passes through the point $P(3, 1)$.

We proceed by approximations. We choose a point *on the parabola* that is nearby $(3, 1)$ and use it to approximate the slope of the tangent line. Let's draw a sketch.



Choose a point close to $P(3, 1)$, say $Q(4, -8)$. Then the slope of the line passing between PQ is:

$$m_{PQ} = \frac{y_Q - y_P}{x_Q - x_P} = \frac{-8 - 1}{4 - 3} = \frac{-9}{1} = -9.0$$

We can consider this a first approximation to the actual slope we want, m , the slope of the tangent line.

Show Animation online.

If we now move the point Q towards P , we can construct the following tables of values:

x in $Q(x, -3x^2 + 12x - 8)$	m_{PQ}	x in $Q(x, -3x^2 + 12x - 8)$	m_{PQ}
4	-9.0	2	-3
3.5	-7.5	2.5	-4.5
3.1	-6.3	2.9	-5.7
3.01	-6.03	2.99	-5.87
3.001	-6.003	2.999	-5.997

What do you think the slope at $P(3, 1)$ would be? Answer: We guess that $m = -6$.

We can then use the point-slope form of the equation of a line:

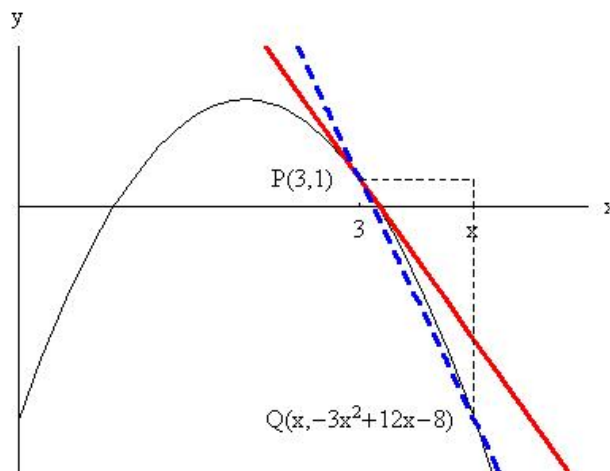
$$y - y_0 = m(x - x_0)$$

to write the equation of the tangent line passing through $(3, 1)$ as:

$$y = -6x + 19.$$

But this was only a guess for the slope—we did not calculate it in a mathematically rigorous manner.

Let's revisit our picture, and see what else we can learn. Let's redraw it without labelling the point Q for a specific value of x , leaving it instead as a function of x .



We say the slope of the tangent line is the **limit** of the slopes of the secant lines, and we express this as:

$$\lim_{Q \rightarrow P} m_{PQ} = m,$$

and, using the equations for the distances:

$$m = \lim_{Q \rightarrow P} \frac{y_Q - y_P}{x_Q - x_P}$$

$$m = \lim_{x \rightarrow 3} \frac{-3x^2 + 12x - 8 - 1}{x - 3} = \lim_{x \rightarrow 3} \frac{-3x^2 + 12x - 9}{x - 3}.$$

What we will be concerned with in the next few days is developing techniques which allow us to calculate quantities like m , and show in a mathematically rigorous fashion, without relying on tables or guesses, that

$$m = \lim_{x \rightarrow 3} \frac{-3x^2 + 12x - 9}{x - 3} = -6.$$

Notice that if we just substituted 3 into $\frac{-3x^2 + 12x - 9}{x - 3}$ we would get $\frac{0}{0}$, an indeterminate form (we will discuss these in Section 2.3).

The Velocity Problem

In our macroscopic world, the velocity of any moving object is not constant, it is always changing due to some force that is acting upon it. The question we ask is “What is the instantaneous velocity of an object”?

Consider an object moving in a gravitational field. It’s height is given by a function $s(t)$, where the specific form of the function is given to us by physicists.

The average velocity over a time interval $[t_1, t_2]$ is given by the rate of change of distance divided by the time:

$$\text{average velocity} = \frac{\text{distance traveled}}{\text{elapsed time}} = \frac{s(t_2) - s(t_1)}{t_2 - t_1}.$$

The instantaneous velocity requires us to know the velocity at a single time, and we can construct a limiting procedure using the average velocity to determine it.

$$\text{instantaneous velocity} = \lim_{t_2 \rightarrow t_1} \frac{s(t_2) - s(t_1)}{t_2 - t_1}.$$

Example 2.1.6 An arrow is shot upward on the moon, and its height is given by the equation $s(t) = 58t - 0.83t^2$, where s is in meters and t is in seconds. Estimate the instantaneous velocity at $t = 1$ s.

SOL: We get the instantaneous velocity by using a limiting procedure on the average velocity. Let $t_1 = 1$.

$$\text{average velocity} = \frac{s(t_2) - s(1)}{t_2 - 1} = \frac{58t_2 - 0.83t_2^2 - 57.17}{t_2 - 1}$$

Notice that substituting $t_2 = 1$ directly in the above gives $\frac{0}{0}$ again.

Let’s construct a table for different values of t_2 :

elapsed time interval $[t_1, t_2]$	average velocity
[1, 2]	55.51
[1, 1.5]	55.925
[1, 1.1]	56.257
[1, 1.01]	56.3317
[1, 1.001]	56.3392

So it appears that as we shorten the time interval, the average velocity is becoming closer to 56.34 m/s, so we guess the instantaneous velocity is 56.34 m/s at $t = 1$ s.

Conclusion

In both cases (tangent line and instantaneous velocity) we are taking a limit as the denominator goes to zero, and the numerator is also zero when this happens.

So you can see how important limits can be. We will spend the next few days looking at limits. Limits are the basis of derivatives, and intimately connected with the use of calculus. With the study of limits, we have begun our study of calculus, and I can not stress enough how important understanding limits is to understanding the ideas of higher mathematics.