The notation is crucial here, so I prepared this handout to supplement our lecture today.

We need to understand some concepts in general terms before we turn our attention to learning techniques to evaluate limits.

The Limit of a Function

Definition We write

$$\lim_{x \to a} f(x) = L$$

and say "the limit of $f(x)$, as x approaches a, equals L", if we can make the values of $f(x)$ arbitrarily close to L by taking x to be sufficiently close to a, but not equal to a.

This means that the values of $f(x)$ become closer and closer to L as we take x closer and closer to a BUT NOT EQUAL TO a. This last is very important. You can also write

$$f(x) \to L \text{ as } x \to a$$

The $x \neq a$ part of the definition is very important. The implications of this are best shown graphically:

In all cases, $\lim\limits_{x \to a} f(x) = L$.

Example Use *Mathematica* to plot $f(t) = \dfrac{\sqrt{t^2+9}-3}{t^2}$. and estimate $\lim\limits_{t \to 0} f(t)$ from the sketch.

```
f[t_] = (Sqrt[t^2 + 9] - 3)/t^2
Plot[f[t], {t, -0.1, 0.1}]
Plot[f[t], {t, -10^(-2), 10^(-2)}]
Plot[f[t], {t, -10^(-4), 10^(-4)}]
```

Example Using *Mathematica*, construct a table of values and graph to guess $\lim_{x \to 0} \frac{\sin x}{x}$.

```
g[x_] = Sin[x]/x
g[x] /. x -> {1.0, 0.5, 0.4, 0.3, 0.2, 0.1, 0.05, 0.01, 0.005, 0.001}
Plot[g[x], {x, -1, 1}]
```

Example Using *Mathematica*, construct a table of values and graph to guess $\lim_{x \to 0} \tan \frac{1}{x}$.

```
h[x_] = Tan[1/x]
h[x] /. x -> {1.0, 1/3, 0.1, 0.5, 0.25, 0.01}
Plot[h[x], {x, -1, 1}]
```

The value of $\tan(1/x)$ oscillates rapidly between $-\infty$ and ∞ as x approaches 0, and so this function does not have a limit as $x \to 0$. We say the limit **does not exist**.

Example Use *Mathematica* to plot $f(x) = \begin{cases} x^2 & x < 0 \\ x+1 & x \geq 0 \end{cases}$ and estimate $\lim_{x \to 0} f(x)$ from the sketch.

```
f[x_] = Piecewise[{{x^2, x < 0}, {x + 1, x >= 0}}]
Plot[f[x], {x, -1, 1}]
```

Conclusions

⇒ Guessing at limits can give incorrect results. It is easy to guess wrong if we pick the wrong x values to evaluate at, and it is hard to know when to stop evaluating, and when numerical round off error is making the results nonsense.

⇒ We need to have a way to indicate if a limit is coming from the right or left.

One Sided Limits

Definition: Right-Hand Limit We write $\lim_{x \to a^+} f(x) = L$ and say

the right-hand limit of $f(x)$ as x approaches a

or

the limit of $f(x)$ as x approaches a from the right

is equal to L if we can make the values of $f(x)$ arbitrarily close to L by taking x to be sufficiently close to a and x is greater than a.

Definition: Left-Hand Limit We write $\lim_{x \to a^-} f(x) = L$ and say

the left-hand limit of $f(x)$ as x approaches a

or

the limit of $f(x)$ as x approaches a from the left

is equal to L if we can make the values of $f(x)$ arbitrarily close to L by taking x to be sufficiently close to a and x is less than a.

These two definitions allow us to write the following statement about limits:

$$\lim_{x \to a} f(x) = L \text{ if and only if } \lim_{x \to a^-} f(x) = L \text{ and } \lim_{x \to a^+} f(x) = L$$

Example

$\lim_{x \to 2^-} g(x) =$

$\lim_{x \to 2^+} g(x) =$

$\lim_{x \to 2} g(x) =$

$g(2) =$

$\lim_{x \to 5^-} g(x) =$

$\lim_{x \to 5^+} g(x) =$

$\lim_{x \to 5} g(x) =$

$g(5) =$

Infinite Limits

NOTE: This mean the <u>limit</u> goes to infinity, not that $x \to \infty$!

Example: Find $\lim_{x\to 1} \frac{1}{(x-1)^2}$.

Solution: As x becomes close to 1 (from either the left or right), $\frac{1}{(x-1)^2}$ grows larger and larger. So we do not approach a number L that would be our limit. Therefore the limit **does not exist.**

However, as $x \to 1$, the function $\frac{1}{(x-1)^2}$ becomes arbitrarily large, so we write this as $\lim_{x\to 1} \frac{1}{(x-1)^2} = \infty$.

> NOTE: This DOES NOT mean that ∞ is a number! What it means is that our function grows without bound, and the limit DOES NOT EXIST. It expresses *the way* in which the limit does not exist.

Definition Let f be a function defined on both sides of a, except possibly at a itself. Then

$$\lim_{x\to a} f(x) = \infty$$

means that the values of $f(x)$ can be made arbitrarily large by taking x sufficiently close to a, but not equal to a.

Definition Let f be a function defined on both sides of a, except possibly at a itself. Then

$$\lim_{x\to a} f(x) = -\infty$$

means that the values of $f(x)$ can be made arbitrarily large negative by taking x sufficiently close to a, but not equal to a.

We say arbitrarily large negative rather than arbitrarily small, because the latter can be a bit confusing. Small usually refers to a number near 0.

We have similar definitions for the one-sided infinite limits, which require we approach only for one side.

Example Plot $k(x) = 1/(x^2-4)$ around $x = 2$ and see what the function looks like.

```
k[x_] = 1 / (x^2-4)
Plot[k[x],{x,1,3}]
```

We can say $\lim_{x\to 2^+} \frac{1}{x^2-4} = \infty$, $\lim_{x\to 2^-} \frac{1}{x^2-4} = -\infty$, and $\lim_{x\to 2} \frac{1}{x^2-4}$ does not exist.

Definition The line $x = a$ is called a vertical asymptote of the curve $y = f(x)$ if at least one of the following statements is true (general case is followed by a specific example):

$$\lim_{x\to a} f(x) = \infty \qquad \lim_{x\to a^-} f(x) = \infty \qquad \lim_{x\to a^+} f(x) = \infty$$

$$\lim_{x\to 1} \frac{1}{(x-1)^2} = \infty \qquad \lim_{x\to 1^-} \frac{1}{1-x} = \infty \qquad \lim_{x\to 1^+} \frac{1}{x} = \infty$$

$$\lim_{x\to a} f(x) = -\infty \qquad \lim_{x\to a^-} f(x) = -\infty \qquad \lim_{x\to a^+} f(x) = -\infty$$

$$\lim_{x\to -1} \frac{x}{(x+1)^2} = -\infty \qquad \lim_{x\to 1^-} \frac{1}{x} = -\infty \qquad \lim_{x\to 0^+} \ln x = -\infty$$

Example MMA can evaluate limits for you, but for the moment you should mainly use it as a check, not to do the work for you!

```
Limit[x/(x + 1)^2, x -> -1]
```