# 1101 Calculus I Lecture 2.6: Horizontal Asymptotes

**Note:** We will not be using the precise definition of limits in this course.

<u>Vertical asymptotes</u> are when we let $x$ approach a number and the function $f$ becomes arbitrarily large (positive or negative).

For example, $\lim\limits_{x \to 0^+} \ln x = -\infty$ means $y = \ln x$ has a vertical asymptote at $x = 0$.

<u>Horizontal asymptotes</u> are when we let $x$ become arbitrarily large (positive or negative) and the function $f$ approaches a number.

This is the more formal way to treat end behaviour, and allows us to examine more complicated functions than we have in the past (before we focussed mostly on rational functions, and said things like "$4x^5 - 3x + 1$ is dominated by $4x^5$ when $x$ is large").

**Definition** Let $f$ be a function defined on some interval $(a, \infty)$. Then $\lim\limits_{x \to \infty} f(x) = L$ means that the values of $f(x)$ can be made arbitrarily close to $L$ by taking $x$ sufficiently large.
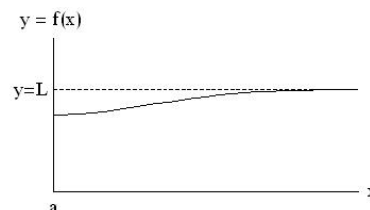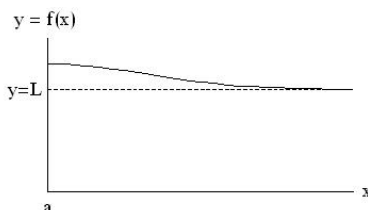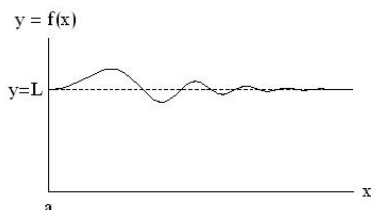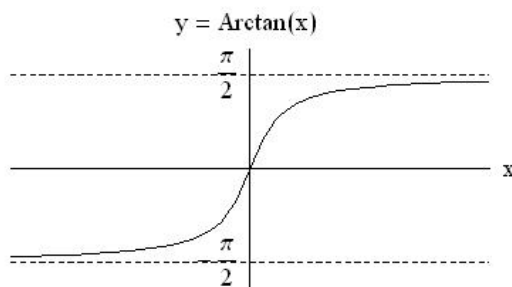


**Definition** Let $f$ be a function defined on some interval $(-\infty, a)$. Then $\lim\limits_{x \to -\infty} f(x) = L$ means that the values of $f(x)$ can be made arbitrarily close to $L$ by taking $x$ sufficiently large negative.
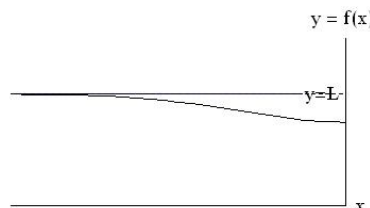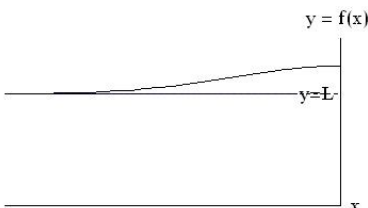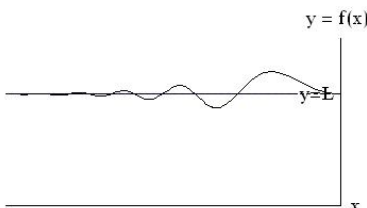


**Definition** The line $y = L$ is called a <u>horizontal asymptote</u> of the curve $y = f(x)$ if either

$$\lim_{x \to \infty} f(x) = L, \quad \text{or} \quad \lim_{x \to -\infty} f(x) = L.$$

**For example**, the sketch of $f(x) = \tan^{-1} x$ shows two horizontal asymptotes:



From the sketch, we see $\lim\limits_{x \to \infty} \arctan x = \dfrac{\pi}{2}$, and $\lim\limits_{x \to -\infty} \arctan x = -\dfrac{\pi}{2}$.

Therefore, the horizontal asymptotes of $f(x) = \arctan(x)$ are $y = \pi/2$, and $y = -\pi/2$.

**Theorems**

If $r > 0$ is a rational number, then $\lim\limits_{x \to \infty} \dfrac{1}{x^r} = 0$.

If $r > 0$ is a rational number such that $x^r$ is defined for all $x$, then $\lim\limits_{x \to -\infty} \dfrac{1}{x^r} = 0$.

> **Technique:** To evaluate a limit at infinity of a rational function or some algebraic functions, where both numerator and denominator are becoming large, we divide the numerator and denominator by the highest power of $x$ in the denominator so we can use the two results above. We don't have to worry about division by zero since we are looking at very large $x$.

Note that $\dfrac{\infty}{\infty}$ is yet another <u>indeterminant form</u>.

**Example** Evaluate $\lim\limits_{x \to \infty} \dfrac{x^2 + 1}{2x^2 - 3x - 2}$.

$$\lim_{x \to \infty} \frac{x^2 + 1}{2x^2 - 3x - 2} \quad \rightarrow \quad \frac{\infty}{\infty} \text{ indeterminant form, divide by largest power of } x \text{ in denominator}$$

$$= \lim_{x \to \infty} \frac{x^2 + 1}{2x^2 - 3x - 2} \cdot \frac{\frac{1}{x^2}}{\frac{1}{x^2}}$$

$$= \lim_{x \to \infty} \frac{1 + \frac{1}{x^2}}{2 - 3\frac{1}{x} - 2\frac{1}{x^2}}$$

$$= \frac{1 + 0}{2 - 3(0) - 2(0)}$$

$$= \frac{1}{2}$$

**Example** Evaluate $\lim\limits_{x \to \infty} (\sqrt{9x^2 + x} - 3x)$.

$$\lim_{x \to \infty} (\sqrt{9x^2 + x} - 3x) \longrightarrow \infty - \infty \quad \text{indeterminant form; need to do some algebra}$$

$$= \lim_{x \to \infty} (\sqrt{9x^2 + x} - 3x) \cdot \frac{(\sqrt{9x^2 + x} + 3x)}{(\sqrt{9x^2 + x} + 3x)}$$

$$= \lim_{x \to \infty} \frac{(9\cancel{x^2} + x - 9\cancel{x^2})}{(\sqrt{9x^2 + x} + 3x)}$$

$$= \lim_{x \to \infty} \frac{x}{(\sqrt{9x^2 + x} + 3x)} \quad \rightarrow \quad \frac{\infty}{\infty} \text{ divide by largest power of } x \text{ in denominator}$$

$$= \lim_{x \to \infty} \frac{x}{(\sqrt{9x^2 + x} + 3x)} \cdot \frac{1/x}{1/x}$$

$$= \lim_{x \to \infty} \frac{1}{\sqrt{9x^2 + x}(\frac{1}{\sqrt{x^2}}) + 3} \quad \text{since } x = \sqrt{x^2} \text{ when } x > 0$$

$$= \lim_{x \to \infty} \frac{1}{\sqrt{9 + \frac{1}{x}} + 3}$$

$$= \frac{1}{\sqrt{9 + 0} + 3}$$

$$= \frac{1}{6}$$

**Example** Find the horizontal asymptotes of the function: $f(x) = \dfrac{\sqrt{2x^2 + 1}}{3x - 5}$.

$$
\begin{aligned}
\lim_{x \to \infty} \frac{\sqrt{2x^2 + 1}}{3x - 5} &= \longrightarrow \frac{\infty}{\infty} \quad \text{direct substitution does not work; indeterminant form} \\
&= \lim_{x \to \infty} \frac{\sqrt{2x^2 + 1}}{3x - 5} \cdot \frac{1/x}{1/x} \\
&= \lim_{x \to \infty} \frac{\sqrt{2x^2 + 1}}{3x - 5} \cdot \frac{1/\sqrt{x^2}}{1/x} \quad \text{since } x = \sqrt{x^2} \text{ for } x > 0 \\
&= \lim_{x \to \infty} \frac{\sqrt{2 + 1/x^2}}{3 - 5/x} \\
&= \frac{\sqrt{2 + 0}}{3 - 0} \\
&= \frac{\sqrt{2}}{3}
\end{aligned}
$$

So $y = \sqrt{2}/3$ is a horizontal asymptote.

$$
\begin{aligned}
\lim_{x \to -\infty} \frac{\sqrt{2x^2 + 1}}{3x - 5} &= \longrightarrow \frac{\infty}{-\infty} \quad \text{direct substitution does not work; indeterminant form} \\
&= \lim_{x \to -\infty} \frac{\sqrt{2x^2 + 1}}{3x - 5} \cdot \frac{1/x}{1/x} \\
&= \lim_{x \to -\infty} \frac{\sqrt{2x^2 + 1}}{3x - 5} \cdot \frac{1/(-\sqrt{x^2})}{1/x} \quad \text{since } x = -\sqrt{x^2} \text{ for } x < 0 \\
&= -\lim_{x \to -\infty} \frac{\sqrt{2 + 1/x^2}}{3 - 5/x} \\
&= -\frac{\sqrt{2 + 0}}{3 - 0} \\
&= -\frac{\sqrt{2}}{3}
\end{aligned}
$$

So $y = -\sqrt{2}/3$ is a horizontal asymptote.

$$
\boxed{\text{Recall: } \lim_{x \to \infty} e^x = \infty \text{ and } \lim_{x \to -\infty} e^x = 0.}
$$

**Example** Evaluate $\displaystyle \lim_{x \to \infty} \frac{1 - e^x}{1 + 2e^x}$.

$$
\begin{aligned}
\lim_{x \to \infty} \frac{1 - e^x}{1 + 2e^x} &\to \frac{-\infty}{\infty} \quad \text{indeterminant form; divide everything by } e^x, \text{ which is same as multiplying by } e^{-x} \\
&= \lim_{x \to \infty} \frac{1 - e^x}{1 + 2e^x} \cdot \frac{e^{-x}}{e^{-x}} \\
&= \lim_{x \to \infty} \frac{e^{-x} - 1}{e^{-x} + 2} \\
&= \frac{0 - 1}{0 + 2} = \frac{-1}{2}
\end{aligned}
$$

Note that $\displaystyle \lim_{x \to -\infty} \frac{1 - e^x}{1 + 2e^x} = \frac{1 - 0}{1 + 2(0)} = 1$.

**Example** Evaluate $\lim\limits_{x\to\infty} \dfrac{\sqrt{9x^6 - x}}{x^3 + 1}$.

$$\lim_{x\to\infty} \frac{\sqrt{9x^6 - x}}{x^3 + 1} \to \frac{\infty}{\infty} \text{ indeterminant form}$$

$$= \lim_{x\to\infty} \frac{\sqrt{9x^6 - x}}{x^3 + 1} \cdot \frac{\frac{1}{x^3}}{\frac{1}{x^3}}$$

$$= \lim_{x\to\infty} \frac{\sqrt{9x^6 - x}\,\frac{1}{\sqrt{x^6}}}{1 + \frac{1}{x^3}} \text{ since for } x > 0,\ x^3 = \sqrt{x^6}$$

$$= \lim_{x\to\infty} \frac{\sqrt{9 - \frac{1}{x^5}}}{1 + \frac{1}{x^3}}$$

$$= \frac{\sqrt{9 - 0}}{1 + 0}$$

$$= 3$$

**Example** Evaluate $\lim\limits_{x\to-\infty} \dfrac{\sqrt{9x^6 - x}}{x^3 + 1}$.

Note that in this problem, we need $\lim\limits_{x\to-\infty}(9x^6 - x) = \lim\limits_{x\to-\infty} x(9x^5 - 1) = (-\infty)(-\infty) = \infty$.

$$\lim_{x\to-\infty} \frac{\sqrt{9x^6 - x}}{x^3 + 1} \to \frac{\infty}{-\infty} \text{ indeterminant form}$$

$$= \lim_{x\to-\infty} \frac{\sqrt{9x^6 - x}}{x^3 + 1} \cdot \frac{\frac{1}{x^3}}{\frac{1}{x^3}}$$

$$= -\lim_{x\to-\infty} \frac{\sqrt{9x^6 - x}\,\frac{1}{\sqrt{x^6}}}{1 + \frac{1}{x^3}} \text{ since for } x < 0,\ x^3 = -\sqrt{x^6}$$

$$= -\lim_{x\to-\infty} \frac{\sqrt{9 - \frac{1}{x^5}}}{1 + \frac{1}{x^3}}$$

$$= -\frac{\sqrt{9 - 0}}{1 + 0}$$

$$= -3$$

**Example** Evaluate $\lim\limits_{x\to\infty} \dfrac{-4x^7 + 5}{(x^2 - 2)(2x^2 - 1)}$.

$$\lim_{x\to\infty} \frac{-4x^7 + 5}{(x^2 - 2)(2x^2 - 1)} \to \frac{-\infty}{\infty} \text{ indeterminant form}$$

$$= \lim_{x\to\infty} \frac{-4x^7 + 5}{2x^4 - 5x^2 + 2} \cdot \frac{\frac{1}{x^4}}{\frac{1}{x^4}}$$

$$= \lim_{x\to\infty} \frac{-4x^3 + 5\frac{1}{x^4}}{2 - 5\frac{1}{x^2} + 2\frac{1}{x^4}}$$

$$= \frac{-4(\infty) + 5(0)}{2 - 5(0) + 2(0)}$$

$$= -\infty$$

The limit does not exist.