# 1101 Calculus I Lecture 2.7 Derivatives and Rates of Change

Before we did some guessing of slopes of tangent lines based on numerical evidence. Now that we know all about limits, we can be far more precise in how we calculate rates of change, and obtaining quantities like slope of tangent line, instantaneous velocity, and other rates of change. We can also define what a derivative of a function is.

Definition The <u>derivative</u> of a function f at a number a is denoted $f'(a) = \lim\limits_{h \to 0} \dfrac{f(a+h) - f(a)}{h}$ if this limit exists.

Interpretation of Derivative as Slope of the Tangent Line

The following sketch make the definition clear.
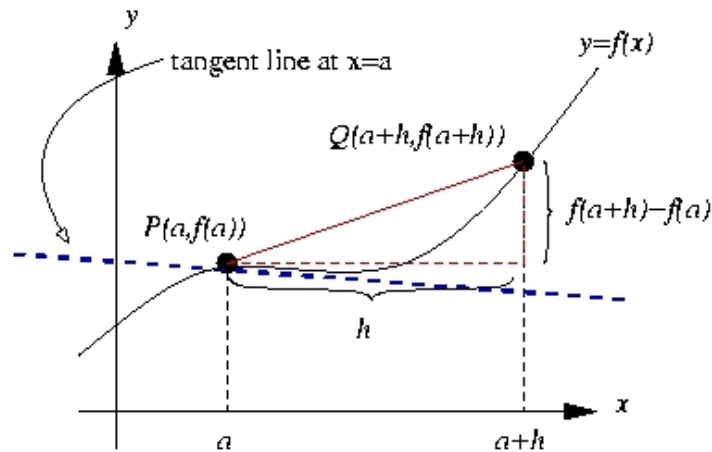
The slope of the line through PQ is $\dfrac{f(a+h) - f(a)}{h}$.

If we take the limit as h approaches 0, the line through PQ approaches the tangent line at P, and therefore the tangent line at P has the slope $m = \lim\limits_{h \to 0} \dfrac{f(a+h) - f(a)}{h}$.
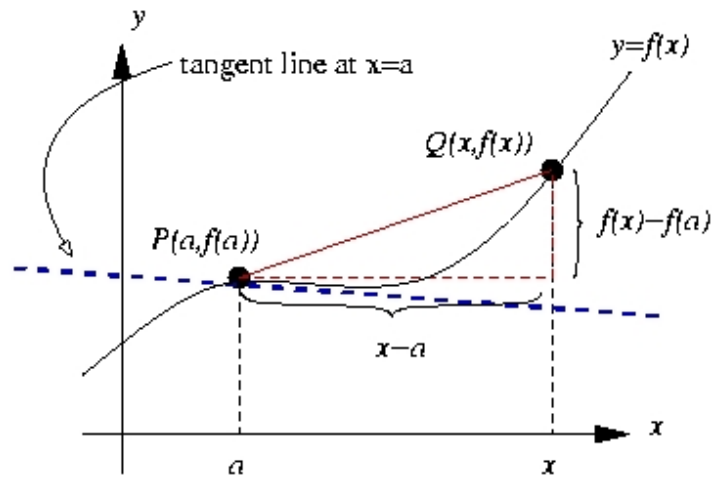
In terms of derivative, we have:

$$
\begin{aligned}
f'(a) &= \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \\
&= \text{slope of tangent line at } x = a \\
&= \text{slope of curve } y = f(x) \text{ at } x = a
\end{aligned}
$$

Definition The <u>tangent line</u> to the curve $y = f(x)$ at the point $P(a, f(a))$ is the line through P with the slope $m = f'(a) = \lim\limits_{h \to 0} \dfrac{f(a+h) - f(a)}{h}$ provided the limit exists.

Alternate way to calculate slope of tangent line

$$m = f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

The slope of the line through PQ is $\dfrac{f(x) - f(a)}{x - a}$.

If we take the limit as x approaches a, the line through PQ approaches the tangent line at P, and therefore the tangent line at P has the slope $m = \lim\limits_{x \to a} \dfrac{f(x) - f(a)}{x - a}$.

Velocities

We are now able to look at the instantaneous velocity question, using our knowledge of limits.

• In general, suppose an object moves along a straight line according to an equation of motion $s = f(t)$, where s is the displacement of the object from the origin at time t.

• The function f is called the position function.

• In the time interval from $t = a$ to $t = a + h$ the change in position is $f(a + h) - f(a)$.

• The average velocity over this interval is $= \dfrac{\text{displacement}}{\text{time interval}} = \dfrac{f(a + h) - f(a)}{h}$.

• As we let $h \to 0$ we obtain the instantaneous velocity,

$$v(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$$

which means the velocity at time $t = a$ is equal to the slope of the tangent line at $t = a$ of f. It is also equal to the derivative $f'(a)$.

Rates of Change in General

The following is how you might see changes represented in experiments for physics or chemistry.

Suppose $y = f(x)$ is the underlying relationship you are studying. Experimentally, you might choose two values of x, say x_1 and x_2, and then measure the value of y corresponding to these values.

If x changes from x_1 to x_2, then the change in x (called the increment in x) is $\Delta x = x_2 - x_1$, which introduces a corresponding change in y, $\Delta y = f(x_2) - f(x_1)$.

The quotient $\dfrac{\Delta y}{\Delta x} = \dfrac{f(x_2) - f(x_1)}{x_2 - x_1}$ is called the average rate of change of y with respect to x over the interval $[x_1, x_2]$.

By analogy with the velocity, we can define an instantaneous rate of change of y with respect to x at $x = x_1$,

$$\text{instantaneous rate of change of } y \text{ with respect to } x \;=\; \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{x_2 \to x_1} \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

A sketch shows that this instantaneous rate of change can also be interpreted as the slope of the tangent line to the curve $y = f(x)$ at $P(x_1, f(x_1))$:
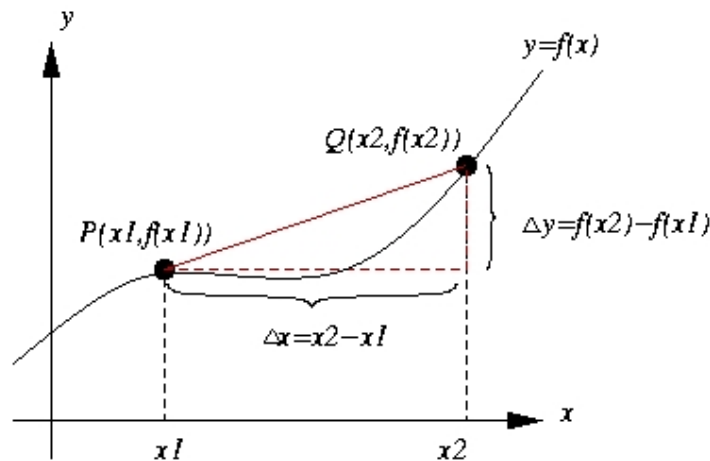
The derivative $f'(a)$ is the instantaneous rate of change of $y = f(x)$ with respect to x when $x = a$.

Example Find the equation of the tangent line to the parabola $y = x^2 - x - 4$ at the point $P(1, -4)$.

Here we have $a = 1$ and $f(x) = x^2 - x - 4$, and we'll use $m = f'(a) = \lim\limits_{x \to a} \dfrac{f(x) - f(a)}{x - a}$.

$$
\begin{aligned}
m = f'(1) &= \lim_{x \to 1} \frac{f(x) - f(1)}{x - 1} \\
&= \lim_{x \to 1} \frac{f(x) - f(1)}{x - 1} \\
&= \lim_{x \to 1} \frac{(x^2 - x - 4) - (-4)}{x - 1} = \lim_{x \to 1} \frac{x(x - 1)}{x - 1} = \lim_{x \to 1} (x) = 1
\end{aligned}
$$

Use the point slope form of the equation of a line:

$$y - y_1 = m(x - x_1)$$

with $m = 1$ and $(x_1, y_1) = P = (1, -4)$ we have the equation for the tangent line:

$$y - (-4) = 1(x - 1) \text{ or } y = x - 5$$

The slope of the tangent line to a curve at a point is sometimes referred to as the slope of the curve at the point. This is because the tangent line approximates the curve at the point.

```
f[x_] = x^2 - x - 4
tangent[x_] = x - 5
Plot[{f[x], tangent[x]}, {x, -5, 5}]
Plot[{f[x], tangent[x]}, {x, 0.5, 1.5}]
Plot[{f[x], tangent[x]}, {x, 0.9, 1.1}]
```

The slope could also be calculated using the alternate formula. Let's do it without specifying the value of a.

$$
\begin{aligned}
m &= \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \\
&= \lim_{h \to 0} \frac{\left((a+h)^2 - (a+h) - 4\right) - \left(a^2 - a - 4\right)}{h} \\
&= \lim_{h \to 0} \frac{1}{h}\left(a^2 + h^2 + 2ah - a - h - 4 - a^2 + a + 4\right) \\
&= \lim_{h \to 0} \frac{1}{h}\left(h^2 + 2ah - h\right) \\
&= \lim_{h \to 0} \frac{h}{h}\left(h + 2a - 1\right) \\
&= \lim_{h \to 0}\left(h + 2a - 1\right) \\
&= 2a - 1
\end{aligned}
$$

We can now use this to have *Mathematica* plot the situation for a variety of values of a, using $y = m(x - a) + f(a)$ as the equation of the tangent line at $x = a$.

```
f[x_] = x^2 - x - 4
m[a_] = 2 a - 1
tangent[x_, a_] = m[a] (x - a) + f[a]
Manipulate[Plot[{f[x], tangent[x, a]}, {x, -5, 5},PlotRange -> {-20, 20}], {a, -2, 2}]
```

At the moment, computing a rate of change just means setting up a derivative formula and then using the techniques we have developed to work out the limits. Lots of examples online.

Notice these techniques are important since if we tried to evaluate the derivative by direct substitution we would get an indeterminant quotient!