Lecture 2.8 Derivative as a Function

In Section 2.7, we saw that we can interpret the derivative at $x = a$ as the slope of the tangent line at $x = a$:

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

which was for a number a.

Now we want to write this as a function of a variable x rather than the number a:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

We now think of $f'(x)$ as a new <u>function</u>, called the derivative of f. We know that the value of the derivative can be interpreted geometrically as the slope of the tangent line to the graph of f at the point $(x, f(x))$.

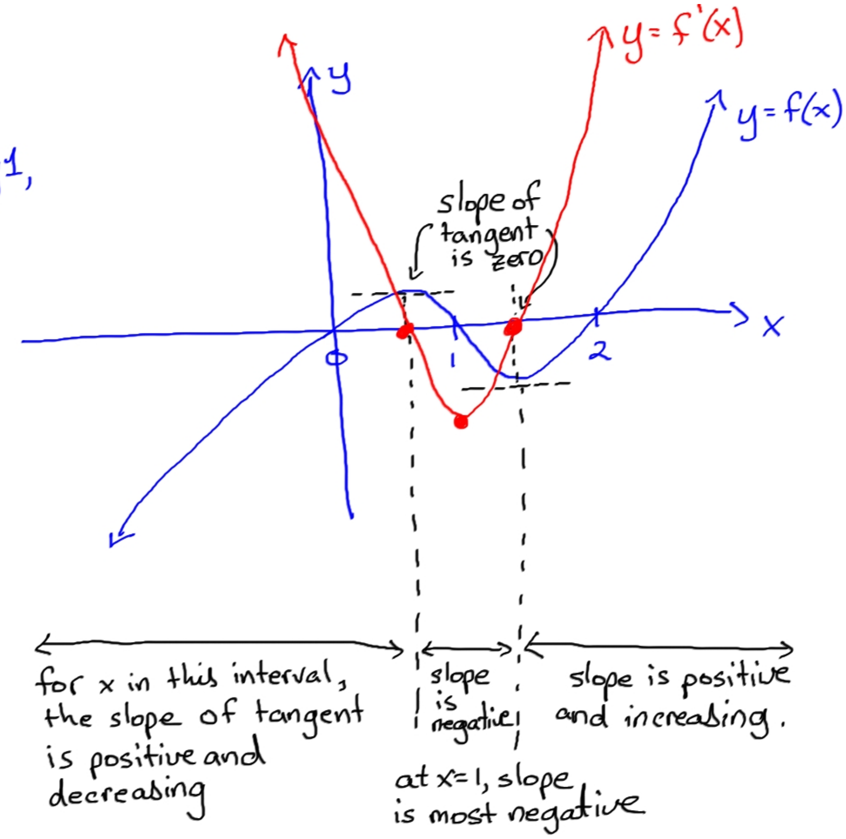
Example Sketch $f(x) = x^3 - 3x^2 + 2x$ and $f'(x)$ without computing $f'(x)$.

Sketch $f(x) = x^3 - 3x^2 + 2x$
$= x(x-1)(x-2)$

Zeroes $x = 0, 1, 2$ all multiplicity 1,
so f changes sign.

$\lim_{x \to \infty} f(x) = \infty$

$\lim_{x \to -\infty} f(x) = -\infty$



Other Common Notations for Derivative of $y = f(x)$

Prime notations:

$$f'(x) = y'$$

Operator notations:

$$f'(x) = \frac{d}{dx}[f(x)] = Df(x) = D_x f(x)$$

Leibniz notations:

$$f'(x) = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx}[f(x)]$$

The following shows where Leibniz notation comes from:

$$\begin{aligned}\frac{dy}{dx} &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\\ &= \lim_{x_2\to x_1}\frac{f(x_2)-f(x_1)}{x_2-x_1}\\ &= \lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}\end{aligned}$$

If we want to indicate the value of a derivative at a specific point, we can write:

$$f'(a) = \left.\frac{dy}{dx}\right|_{x=a}$$

which we read as the "the derivative of y with respect to x evaluated at $x = a$".

Definition A function f is differentiable at a if $f'(a)$ exists. It is differentiable on an open interval if it is differentiable at every number in the interval.

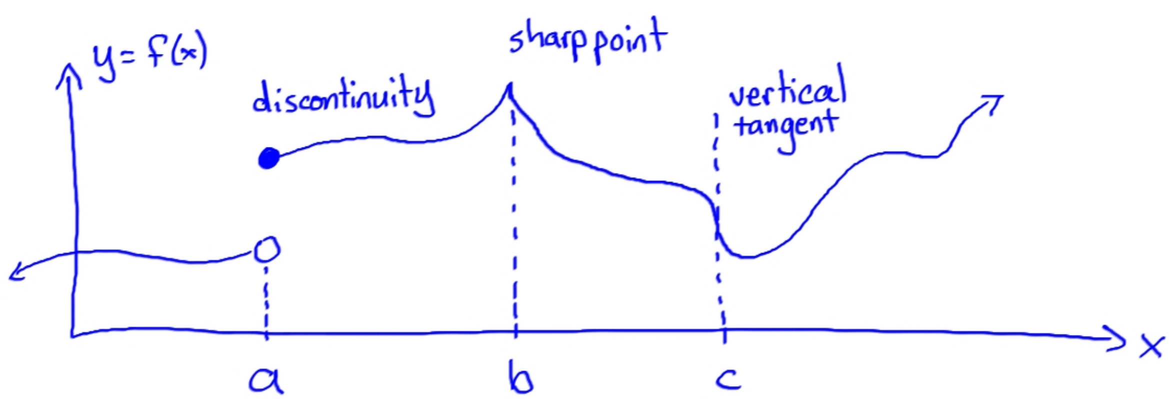
Theorem If f is differentiable at a, then f is continuous at a. Note: The converse is NOT true!! DANGER!! DANGER!!

Example Where is $f(x) = |x|$ differentiable? Answer: everywhere except $x = 0$.

How can a function fail to be differentiable? In general, if the graph of a function has a "corner" or "kink" in it, then the graph has no tangent at this point and so is not differentiable at this point. The left and right limits are different at this point.

If the function is discontinuous, then it is not differentiable.

If the function has a vertical tangent line at $x = a$, then $\lim_{x\to a}|f'(x)| = \infty$, and it is not differentiable at $x = a$.



$f(x)$ is not differentiable at:

- $x = a$ because f is not continuous at $x = a$
- $x = b$ because the limits in derivative from left and right are not equal (an example is $f(x) = |x|$ at $x = 0$)
- $x = c$ because the derivative is infinite (an example is $f(x) = \sqrt{x}$ at $x = 0$)