Linear Approximation

We have already been working with the linear approximation, since it is nothing more than the tangent line to the curve. Let's build up the idea of the linear approximation. Consider the curve $y = \sqrt{x-4}$. What does it look like?

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f[x_] = Sqrt[x-4]
Plot[f[x],{x,4,7}]
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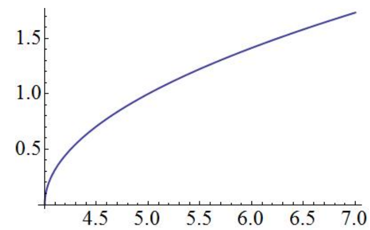

In general, we can express the linear approximation as:

$$\begin{aligned} y - y_0 &= m(x - x_0) \\ y &= y_0 + m(x - x_0) \\ L(x) &= y_0 + m(x - x_0) \\ L(x) &= f(x_0) + f'(x_0)(x - x_0) \end{aligned}$$

We can then say:

$$f(x) \sim f(x_0) + f'(x_0)(x - x_0) \text{ if } x \sim x_0.$$

If we are working about the point $(5, 1)$ in the $f(x) = \sqrt{x-4}$ example, the things we need are:

$$\begin{aligned} f(x_0) = f(5) &= \sqrt{5-4} = 1 \quad \text{We chose } x = 5 \text{ so we could evaluate the square root here!} \\ f'(x) &= \frac{d}{dx}[\sqrt{x-4}] = \frac{1}{2}(x-4)^{-1/2}(1) = \frac{1}{2\sqrt{x-4}} \\ f'(x_0) = f'(5) &= \frac{1}{2\sqrt{5-4}} = \frac{1}{2} \end{aligned}$$

So the linear approximation is:

$$\sqrt{x-4} \sim 1 + \frac{1}{2}(x-5) \text{ if } x \sim 5.$$

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Plot[{f[x], 1 + 1/2 (x - 5)}, {x, 4, 7}]
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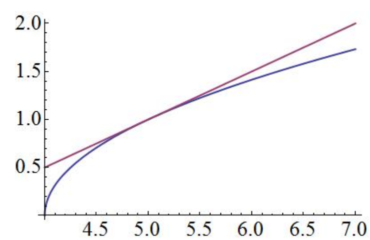

We can see that the linear approximation is an approximation to the curve if we are near $x = x_0$. We can use this to approximate square roots, or other things, depending on what the function is.

For example, say we wanted to approximate $\sqrt{1.08}$. Here is how we would proceed for $\sqrt{1.08}$, using the result from above that

$$\sqrt{x-4} \sim 1 + \frac{1}{2}(x-5) \text{ if } x \sim 5.$$

$$\begin{aligned}
\sqrt{1.08} &= \sqrt{5.08-4} = f(5.08) \\
&\sim L(5.08) = 1 + \frac{1}{2}(5.08-5) \text{ since } 5.08 \sim 5 \\
&\sim 1 + \frac{0.08}{2} \\
&\sim 1.04
\end{aligned}$$

Compare this to the computer result $\sqrt{1.08} = 1.03923$. We can tell from our graph that the linear approximation will be an overestimation since it lies above the actual graph of the function.

Notice that the approximation is around $x_0 \sim 5$, not $x_0 \sim 1$, which you might think you would want when you are working with $\sqrt{1.08}$. You can use different functions to approximate $\sqrt{1.08}$, there was nothing special about our choice above.

Example Let's find the linear approximation to $f(x) = \sin x$ about $x_0 = \frac{\pi}{4}$.

The linear approximation will be:

$$L(x) = f(x_0) + f'(x_0)(x - x_0) = f(\pi/4) + f'(\pi/4)(x - \pi/4)$$

We will need the function and derivative evaluated at $\pi/4$:

$$\begin{aligned}
f(x) &= \sin x \\
f(\pi/4) &= \sin(\pi/4) = \frac{1}{\sqrt{2}} \\
f'(x) &= \cos x \\
f'(\pi/2) &= \cos(\pi/4) = \frac{1}{\sqrt{2}}
\end{aligned}$$

Therefore:

$$\sin x \sim L(x) = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\left(x - \frac{\pi}{4}\right) \text{ if } x \sim \pi/2$$

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Plot[{f[x], 1/Sqrt[2] + 1/Sqrt[2] (x - Pi/4)}, {x, 0, Pi}]
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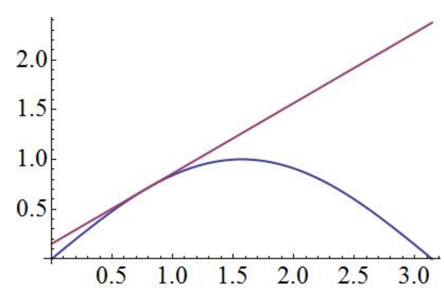
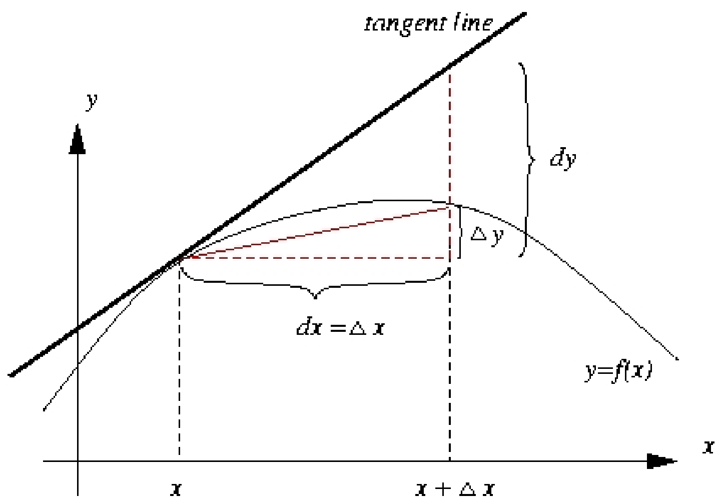


Differentials

Differentials can be thought of as small amount of a quantity; for example, if x is a length, the differential dx is a small amount of length (in the x direction). If V is a volume, then dV is a small amount of volume (in the V direction).

The geometric meaning of differentials is shown in the following diagram, and how they relate to the differences Δx and Δy which we saw before in the definition of derivative. This diagram is essentially the schematics we used in proving derivative rules with a bit more detail.



So, the differential $dx = \Delta x$ can be any real number, and we then define the differential dy as $dy = f'(x)\,dx$. The quantity Δy represents a change in the original curve, and dy represents a change in the tangent line.

From the diagram, it becomes apparent why we use $\frac{dy}{dx}$ to represent the derivative, since $\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = f'(x) = \frac{dy}{dx}$ is the slope of the tangent line at x.