Example $f(x) = x^2 + 3x - 4$, find $f'(x)$.

$$\begin{aligned}
f(x) &= x^2 + 3x - 4 \\
f'(x) &= \frac{d}{dx}[x^2 + 3x - 4] \\
&= \frac{d}{dx}[x^2] + 3\frac{d}{dx}[x] - \frac{d}{dx}[4] \quad \text{Sum Rule} \\
&= 2x^{2-1} + 3(1)x^{1-1} - 0 \quad \text{Power Rule, Constant Rule} \\
&= 2x + 3
\end{aligned}$$

Example $f(x) = x + \frac{1}{x}$, find $f'(x)$.

$$\begin{aligned}
f(x) &= x + \frac{1}{x} \\
&= x + x^{-1} \\
f'(x) &= \frac{d}{dx}[x + x^{-1}] \\
&= \frac{d}{dx}[x] + \frac{d}{dx}[x^{-1}] \quad \text{Sum Rule} \\
&= (1)x^{1-1} + (-1)x^{-1-1} \quad \text{Power Rule} \\
&= 1 - x^{-2} \\
&= 1 - \frac{1}{x^2}
\end{aligned}$$

Example $f(x) = (x^2 + 4x + 3)/\sqrt{x}$, find $f'(x)$.

$$\begin{aligned}
f(x) &= \frac{x^2}{x^{1/2}} + 4\frac{x}{x^{1/2}} + 3\frac{1}{x^{1/2}} \\
&= x^{2-1/2} + 4x^{1-1/2} + 3x^{-1/2} \\
&= x^{3/2} + 4x^{1/2} + 3x^{-1/2} \\
f'(x) &= \frac{d}{dx}[x^{3/2} + 4x^{1/2} + 3x^{-1/2}] \\
&= \frac{d}{dx}[x^{3/2}] + 4\frac{d}{dx}[x^{1/2}] + 3\frac{d}{dx}[x^{-1/2}] \quad \text{Sum Rule} \\
&= \frac{3}{2}x^{3/2-1} + 4\frac{1}{2}x^{1/2-1} + 3\frac{-1}{2}x^{-1/2-1} \quad \text{Power Rule} \\
&= \frac{3}{2}x^{1/2} + 2x^{-1/2} - \frac{3}{2}x^{-3/2}
\end{aligned}$$

Example Find the points on the curve $y = f(x)$ where the tangent line is horizontal: $f(x) = x^4 - 6x^2 + 4$.

The derivative is equal to the slope of the tangent line.
If the tangent line is horizontal, the slope must by zero.
Therefore, we want to solve $f'(a) = 0$ for a.
There may be more than one such a.
Then, the points on the curve will be $(a, f(a))$.

$$f(x) = x^4 - 6x^2 + 4$$

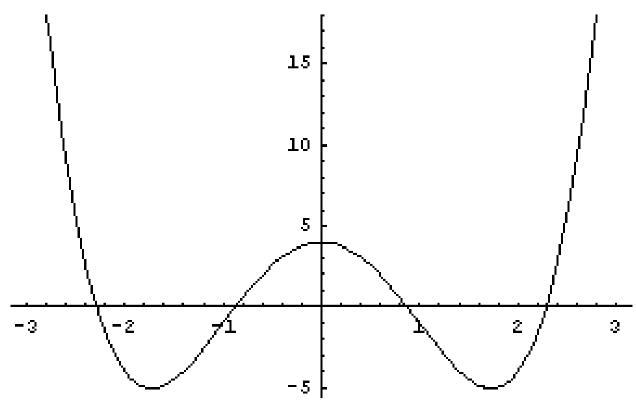
$$
\begin{aligned}
f'(x) &= \frac{d}{dx}[x^4 - 6x^2 + 4] \\
&= \frac{d}{dx}[x^4] - 6\frac{d}{dx}[x^2] + \frac{d}{dx}[4] \quad \text{Sum, Difference, Constant Multiple Rules} \\
&= 4x^{4-1} - 6(2)x^{2-1} + 0 \quad \text{Power Rule, Constant Rule} \\
f'(x) &= 4x^3 - 12x \\
f'(a) = 0 &\rightarrow 4a^3 - 12a = 0 \\
& 4a(a^2 - 3) = 0 \\
4a = 0 &\quad \text{or} \quad (a^2 - 3) = 0 \\
a = 0 &\quad \text{or} \quad a = \pm\sqrt{3}
\end{aligned}
$$

There are three points where the tangent is horizontal:

$$
\begin{aligned}
&(0, f(0)) = (0, 4) \\
&(\sqrt{3}, f(\sqrt{3})) = (\sqrt{3}, -5) \\
&(-\sqrt{3}, f(-\sqrt{3})) = (-\sqrt{3}, -5)
\end{aligned}
$$

You can get a plot using *Mathematica* if you like, and verify that the tangent is horizontal at these three points:

```
Plot[x^4 - 6x^2 + 4, {x, -3, 3}]
```



Example Find the equation of the tangent line to the curve $y = x\sqrt{x}$ at the point $(1, 1)$.

The derivative is equal to the slope of the tangent line.
Therefore, we want to find $f'(x)$.
The point we are interested in is $(1, 1)$, which means $x = 1$.
The slope of the tangent line at $x = 1$ is $f'(1)$.
The tangent line goes through the point $(1, 1)$.
The equation of the tangent line can be found from $y - y_0 = m(x - x_0)$.
The equation of the tangent line will be $y - 1 = f'(1)(x - 1)$.

$$
f(x) = x\sqrt{x}
$$

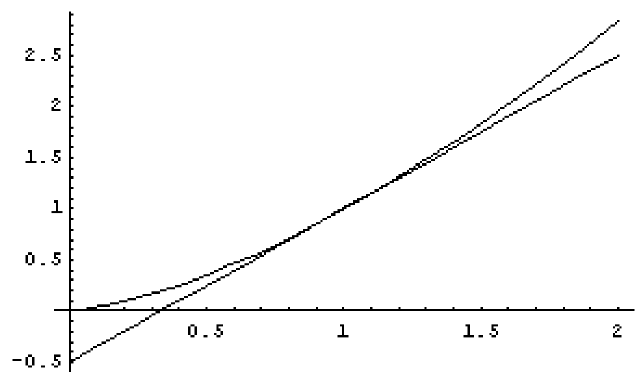
$$\begin{aligned}
&= x \cdot x^{1/2} = x^{3/2} \\
f'(x) &= \frac{d}{dx}[x^{3/2}] \\
&= \frac{3}{2}x^{3/2-1} \quad \text{Power Rule} \\
&= \frac{3}{2}x^{1/2} \\
f'(1) &= \frac{3}{2}
\end{aligned}$$

The equation of the tangent line is therefore:

$$\begin{aligned}
y - 1 &= \frac{3}{2}(x-1) \\
y &= \frac{3}{2}x - \frac{1}{2}
\end{aligned}$$

Mathematica can help you visualize the situation:

```
Plot[{x*Sqrt[x], 3*x/2 - 1/2}, {x, 0, 2}]
```



Example At what point on the curve $y = e^x + x$ is the tangent line parallel to the line $y = 2x$?

Let $f(x) = e^x + x$.
Two lines are parallel if the have the same slope.
The slope of $y = 2x$ is $m = 2$.
The slope of the tangent to the curve at $x = a$ is the derivative $f'(a)$.
Therefore, we want to solve the equation $f'(a) = 2$ for all possible values of a.
The point on the curve where the tangent line is parallel to the curve $y = 2x$ will be $(a, f(a))$.

$$\begin{aligned}
f'(x) &= \frac{d}{dx}[e^x + x] \\
&= \frac{d}{dx}[e^x] + \frac{d}{dx}[x] \quad \text{Sum Rule} \\
&= e^x + (1)x^{1-1} \quad \text{Exponential Rule, Power Rule} \\
&= e^x + 1
\end{aligned}$$

$$\begin{aligned} f'(a) &= e^a + 1 \\ f'(a) = 2 \quad \to& \quad e^a + 1 = 2 \\ & \quad e^a = 1 \\ & \quad \ln(e^a) = ln1 \\ & \quad a = 0 \end{aligned}$$

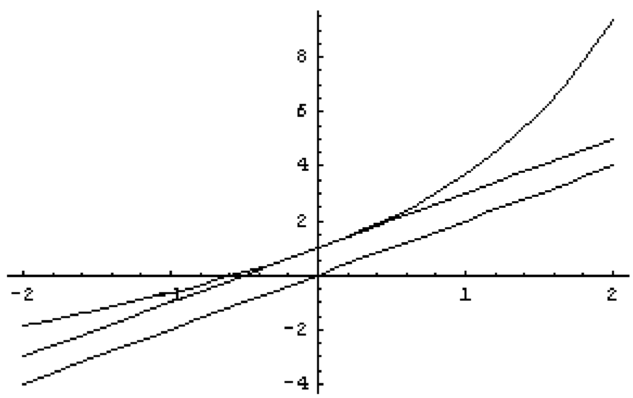
The point where the tangent line is parallel to $y = 2x$ is $(a, f(a)) = (0, e^0 + 0) = (0, 1)$.

The equation of the tangent line is

$$\begin{aligned} y - y_0 &= m(x - x_0) \\ y - y_0 &= f'(x_0)(x - x_0) \\ y - 1 &= 2(x - 0) \\ y &= 2x + 1 \end{aligned}$$

Verify graphically using *Mathematica*.

```
Plot[{Exp[x] + x, 2x, 2x + 1}, {x, -2, 2}]
```



Example $f(x) = e^x/(1 + x^2)$, find $f'(x)$.

$$\begin{aligned} f'(x) &= \frac{d}{dx}\left[\frac{e^x}{1+x^2}\right] \\ &= \frac{(1+x^2)\frac{d}{dx}[e^x] - e^x\frac{d}{dx}[1+x^2]}{(1+x^2)^2} \quad \text{Quotient Rule} \\ &= \frac{(1+x^2)e^x - e^x(0 + 2x^{2-1})}{(1+x^2)^2} \quad \text{Exponential Rule, Constant Rule, Power Rule} \\ &= \frac{(1+x^2)e^x - e^x(2x)}{(1+x^2)^2} \\ &= \frac{e^x(x^2 - 2x + 1)}{(1+x^2)^2} \\ &= \frac{e^x(x-1)^2}{(1+x^2)^2} \end{aligned}$$

Example $y = (x^2 - 2x)e^x$, find y'.

$$\begin{aligned}
y &= (x^2 - 2x)e^x \\
y' &= \frac{d}{dx}[(x^2 - 2x)e^x] \\
&= \frac{d}{dx}[(x^2 - 2x)]e^x + \frac{d}{dx}[e^x](x^2 - 2x) \quad \text{Product Rule} \\
&= (2x^{2-1} - 2(1)x^{1-1})e^x + e^x(x^2 - 2x) \quad \text{Power Rule, Exponential Rule} \\
&= (2x - 2)e^x + e^x(x^2 - 2x) \\
&= e^x(x^2 - 2)
\end{aligned}$$

Example $y = (x - 2)/(x - 1)$, find y'.

$$\begin{aligned}
y &= (x-2)/(x-1) \\
y' &= \frac{d}{dx}\left[(x-2)/(x-1)\right] \\
&= \frac{(x-1)\frac{d}{dx}[x-2] - (x-2)\frac{d}{dx}[x-1]}{(x-1)^2} \quad \text{Quotient Rule} \\
&= \frac{(x-1)((1)x^{1-1} - 0) - (x-2)((1)x^{1-1} - 0)}{(x-1)^2} \quad \text{Power Rule, Constant Rule} \\
&= \frac{(x-1)(1) - (x-2)(1)}{(x-1)^2} \\
&= \frac{x - 1 - x + 2}{(x-1)^2} \\
&= \frac{1}{(x-1)^2}
\end{aligned}$$

Example Find equations of the tangent lines to the curve $y = (x - 1)/(x + 1)$ that are parallel to the line $x - 2y = 2$.

Let $f(x) = (x - 1)/(x + 1)$.
Two lines are parallel if the have the same slope.
$x - 2y = 2$ is the same as $y = x/2 - 1$.
The slope of $y = x/2 - 1$ is $1/2$.
The slope of the tangent to the curve at $x = a$ is the derivative $f'(a)$.
Therefore, we want to solve the equation $f'(a) = 1/2$ for all possible values of a.
The points on the curve where the tangent line is parallel to the curve $y = x/2 - 1$ will be $(a, f(a))$.
There may be more than one such point.
The equation of the tangent lines will be given by $y - f(a) = \frac{1}{2}(x - a)$.

$$\begin{aligned}
f'(x) &= \frac{(x+1)\frac{d}{dx}(x-1) - (x-1)\frac{d}{dx}(x+1)}{(x+1)^2} \\
&= \frac{(x+1)(1) - (x-1)(1)}{(x+1)^2}
\end{aligned}$$

$$\begin{aligned} &= \frac{2}{(x+1)^2} \\ f'(a) &= \frac{2}{(a+1)^2} = \frac{1}{2} \\ & 4 = (a+1)^2 \\ & \pm 2 = a+1 \\ a+1=-2 \quad \text{or} \quad & a+1 = 2 \\ a=-3 \quad \text{or} \quad & a=1 \end{aligned}$$

The equation of the tangents line at $(1, f(1)) = (1, 0)$ is

$$y - 0 = \frac{1}{2}(x-1) \longrightarrow y = \frac{1}{2}(x-1)$$

The equation of the tangents line at $(-3, f(-3)) = (-3, 2)$ is

$$y - 2 = \frac{1}{2}(x+3) \longrightarrow y = \frac{x}{2} + \frac{7}{2}$$

Verify graphically on *Mathematica*:

```
Plot[{(x - 1)/(x + 1), x/2 - 1/2, x/2 + 7/2, x/2 - 1}, {x, -6, 4},PlotRange -> {-5, 5}]
```