1101 Calculus I 3.4 The Chain Rule

Consider $y = \sqrt{x^2+1}$. How would you differentiate it? We can't use any of the current rules.

However, we can rewrite the function using composition: $y = (f \circ g)(x) = f(g(x))$ where $f(x) = \sqrt{x}$, and $g(x) = x^2 + 1$.

We can use our current rules to find the derivatives of f and g!

What we need, and the chain rule provides, is a way of finding the derivative of the composition of functions.

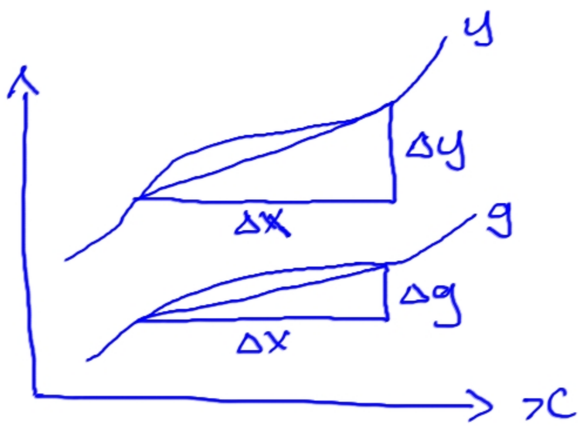
Motivation of the Chain Rule

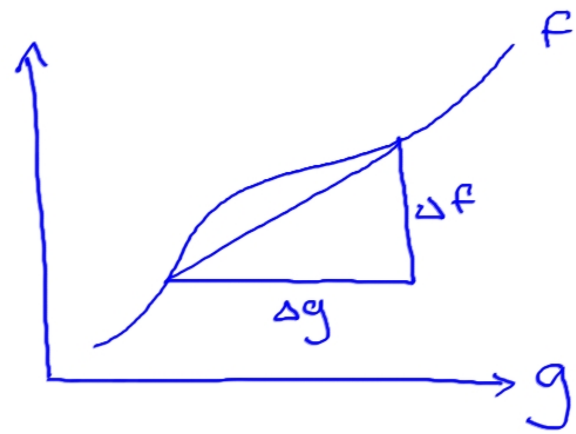
There is a proof in the text, I would rather we looked at a motivation and see why the motivation isn't actually a proof.

Given $y(x) = f(g(x))$, what is $y'(x)$?

Let's try the usual building of $\lim\limits_{\Delta x \to 0} \dfrac{\Delta y}{\Delta x}$.

Notice that y and g depend on x, but f really depends on g, so we need two schematics:





From these schematics, we see that as $\Delta x \to 0$, we have

$$\lim_{\Delta x \to 0} \Delta y = 0$$
$$\lim_{\Delta x \to 0} \Delta g = 0$$
$$\lim_{\Delta x \to 0} \Delta f = \lim_{\Delta g \to 0} \Delta f = 0$$
$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = y'$$
$$\lim_{\Delta x \to 0} \frac{\Delta g}{\Delta x} = g'$$
$$\lim_{\Delta g \to 0} \frac{\Delta f}{\Delta g} = f'$$

So, if x is changed to $x + \Delta x$, y becomes $y + \Delta y$ and f becomes $f + \Delta f$.

So $y = f$ (suppressing all the functional dependence for the moment) becomes $y + \Delta y = f + \Delta f$.

$$
\begin{aligned}
y + \Delta y &= f + \Delta f \\
\Delta y &= f + \Delta f - y \\
\Delta y &= \cancel{f} + \Delta f - \cancel{f} \\
\Delta y &= \Delta f \text{ which isn't surprising if you think about it} \\
\frac{\Delta y}{\Delta x} &= \frac{\Delta f}{\Delta x} \\
\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} &= \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x}
\end{aligned}
$$

Now, the left hand side is just y', but the right hand side is <u>not</u> f'! Remember, f depended on g. The right hand side would be f' if we had $f(x)$ instead of $f(g(x))$.

It might seem like we could do the following:

$$
\begin{aligned}
y' &= \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta g} \frac{\Delta g}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta g} \lim_{\Delta x \to 0} \frac{\Delta g}{\Delta x} \\
&= \lim_{\Delta g \to 0} \frac{\Delta f}{\Delta g} \lim_{\Delta x \to 0} \frac{\Delta g}{\Delta x} \\
&= f' g'
\end{aligned}
$$

which actually is the chain rule! But there is a subtle problem in the relationship between Δx and Δg. As Δx goes to zero we do have Δg going to zero, but it could be that $\Delta g = 0$ when $\Delta x \neq 0$ (this could happen if g is not one-to-one). This is what breaks this "proof" (which is why I called it a motivation instead of a proof). If Δg was equal to zero only when Δx was equal to zero, this would be a proof.

When we proved the product rule and quotient rule (and others) using this technique, we always works with one schematic with x on the horizontal axis, so we didn't run into this problem. Those were proofs.

If you want to see the correct proof of the chain rule, it is in the text. I will not ask you to reproduce a proof of the chain rule on tests.

Remember that in the prime notation for derivative, the prime means *derivative with respect to the independent variable*:

$$
\begin{aligned}
f'(x) &= \frac{d}{dx} f(x) \\
f'(w) &= \frac{d}{dw} f(w) \\
f'(h) &= \frac{d}{dh} f(h) \\
f'(\Box) &= \frac{d}{d\Box} f(\Box) \quad \text{where } \Box \text{ can be anything}
\end{aligned}
$$

What the Chain Rule Does

If $y = f(x)$, the derivative of y is $y' = \dfrac{dy}{dx}$.

Chain Rule using Leibniz notation: If $y = f(u)$ and $u = g(x)$, the derivative of $y' = \dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx}$.

The text shows a way of thinking of the chain rule in terms of "inner" and "outer" functions.

I think the Leibniz notation is the easiest to understand. It expresses the chain rule by simply inserting the number one in the notation:

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

and it allows one to easily create longer chains:
If $y = f(g(h(x)))$ decompose as:

$$\begin{aligned} y &= f(u) \\ u &= g(w) \\ w &= h(x) \end{aligned}$$

and the chain rule now has two links in the chain:

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dw}\frac{dw}{dx}$$

Example Given $y = \sqrt{x^2+1}$, find y'.

Decompose the function:

$$\begin{aligned} y &= \sqrt{u} = u^{1/2} \\ u &= x^2 + 1 \end{aligned}$$

Find the derivative using the chain rule:

$$\begin{aligned} y' &= \frac{dy}{dx} \\ &= \frac{dy}{du}\frac{du}{dx} \text{ the chain rule} \\ &= \frac{d}{du}[u^{1/2}]\frac{d}{dx}[x^2+1] \text{ sub in the quantities} \\ &= (\frac{1}{2\sqrt{u}})(2x) \text{ take the derivative} \\ &= \frac{x}{\sqrt{x^2+1}} \text{ back substitute and simplify} \end{aligned}$$

Example If $y = \sin(x^2)$, find y'.

Decompose:

$$\begin{aligned} y &= \sin u \\ u &= x^2 \end{aligned}$$

Find the derivative using the chain rule:

$$\begin{aligned} y' &= \frac{dy}{dx} \\ &= \frac{dy}{du}\frac{du}{dx} \\ &= \frac{d}{du}[\sin u]\frac{d}{dx}[x^2] \\ &= (\cos u)(2x) \\ &= (\cos(x^2))(2x) \\ &= 2x\cos(x^2) \end{aligned}$$

Example If $y = \sin^2 x$, find y'.

Decompose:

$$y = u^2$$
$$u = \sin x$$

Find the derivative using the chain rule:

$$\begin{aligned} y' &= \frac{dy}{dx} \\ &= \frac{dy}{du}\frac{du}{dx} \\ &= \frac{d}{du}[u^2]\frac{d}{dx}[\sin x] \\ &= (2u)(\cos x) \\ &= 2\sin x \cos x \end{aligned}$$

Example $y = e^{x^2}$, $y' = 2xe^{x^2}$.

Decompose:

$$y = e^u$$
$$u = x^2$$

Apply chain rule:

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du}\frac{du}{dx} \\ &= \frac{d}{du}[e^u]\frac{d}{dx}[x^2] \\ &= (e^u)(2x) = 2xe^{x^2} \end{aligned}$$

Example Use Chain Rule to show why $\dfrac{d}{dx}a^x = a^x \ln a$.

$$\begin{aligned} a^x &= (e^{\ln a})^x = e^{(\ln a)x} = e^u, \quad u = (\ln a)x \\ \frac{d}{dx}a^x &= \frac{d}{dx}e^u = \frac{d}{du}\left[e^u\right]\frac{du}{dx} = e^u \ln a = a^x \ln a \end{aligned}$$

Example $y = \tan(\cos(x^2))$, find dy/dx.

Decompose:

$$y = \tan u$$
$$u = \cos w$$
$$w = x^2$$

Apply chain rule:

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du}\frac{du}{dw}\frac{dw}{dx} \\ &= \frac{d}{du}[\tan u]\frac{d}{dw}[\cos w]\frac{d}{dx}[x^2] \\ &= [\sec^2 u][-\sin w][2x] \\ &= -2x\sec^2(\cos x^2)\sin(x^2) \end{aligned}$$

Example $y = \cos(\sqrt{e^{x^2+2}})$, find dy/dx.

Decompose:

$$\begin{aligned} y &= \cos u \\ u &= \sqrt{v} = v^{1/2} \\ v &= e^w \\ w &= x^2 + 2 \end{aligned}$$

Apply chain rule:

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du}\frac{du}{dv}\frac{dv}{dw}\frac{dv}{dw} \\ &= \frac{d}{du}[\cos u]\frac{d}{dv}[v^{1/2}]\frac{d}{dw}[e^w]\frac{d}{dx}[x^2+2] \\ &= \Big(-\sin u\Big)\Big(\frac{1}{2}v^{-1/2}\Big)\Big(e^w\Big)\Big(2x\Big) \\ &= \Big(-\sin(\sqrt{e^{x^2+2}})\Big)\Big((e^{x^2+2})^{-1/2}\Big)\Big(e^{x^2+2}\Big)\Big(x\Big) \\ &= -x\sqrt{e^{x^2+2}}\sin(\sqrt{e^{x^2+2}}) \end{aligned}$$

Sometimes you don't want to do the full decomposition at the start.

Example If $y = \sin(x^2 + e^{x^3})$, find y'.

Decompose:

$$\begin{aligned} y &= \sin u \\ u &= x^2 + e^{x^3} \end{aligned}$$

We can take the derivative of x^2 with respect to x, but not e^{x^3} (we need another chain rule for it).

$$\begin{aligned} y' &= \frac{dy}{dx} \\ &= \frac{dy}{du}\frac{du}{dx} \\ &= \frac{d}{du}[\sin u]\frac{d}{dx}[x^2 + e^{x^3}] \\ &= (\cos u)(2x + \frac{d}{dx}[e^{x^3}]) \end{aligned}$$

Do the final derivative as an aside:

$$\begin{aligned} \frac{d}{dx}[e^{x^3}] &= \frac{d}{dx}[e^w], \quad w = x^3 \text{ decompose} \\ &= \frac{d}{dw}[e^w]\frac{dw}{dx} \text{ this step is the chain rule} \\ &= \Big(e^w\Big)\Big(3x^2\Big) \text{ evaluate the two derivatives} \\ &= 3x^2e^{x^3} \text{ backsubstitute and simplify} \end{aligned}$$

Finally,

$$\begin{aligned} y' &= (\cos u)(2x + 3x^2e^{x^3}) \\ &= (2x + 3x^2e^{x^3})\cos(x^2 + e^{x^3}) \end{aligned}$$

Once you get some practice, you will be able to do the decompositions on the fly in your head. But explicitly include them as long as they help you get the correct answer. Remember, if you show less work and your answer is incorrect you get less partial credit!

Here is an example with most of the work done in my head, and not explicitly shown. If it is confusing to you, redo the example and include more steps.

Example If $y = \dfrac{x^2+1}{\cos(x^3-1)}$, find y'.

$$\begin{aligned}
y' &= \frac{d}{dx}\left[\frac{x^2+1}{\cos(x^3-1)}\right] \\
&= \frac{\cos(x^3-1)(2x) - (x^2+1)(-\sin(x^3-1)(3x^2))}{\cos^2(x^3-1)} \\
&= \frac{2x\cos(x^3-1) + 3x^2(x^2+1)\sin(x^3-1)}{\cos^2(x^3-1)}
\end{aligned}$$