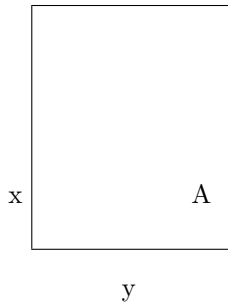# 1101 Calculus: Optimization Problems in $\mathbb{R}^2$ viewed in $\mathbb{R}^3$

Consider the optimization problem in $\mathbb{R}^2$:

Example A farmer has 2400 ft of fencing. What are the dimensions of the rectangular pen that produce the largest area?

We need a rectangle. The rectangle should have maximum area for a given perimeter.

x   A

     y

The perimeter is $P = 2x + 2y$.
The area is $A = xy$. This is what we want to maximize.
We need to eliminate y from the equation for A. Use $P = 2x + 2y = 2400, \longrightarrow y = 1200 - x$.
Therefore, $A = xy = x(1200 - x) = 1200x - x^2$.
If $x < 0$, the area would be negative. This is unphysical.
If $x > 1200$, the area would be negative. This is unphysical.
The domain for the area is $0 \leq x \leq 1200$.

• Find the maximum of $A(x) = 1200x - x^2, 0 \leq x \leq 1200$.

$A' = 1200 - 2x$.
$A' = 0 = 1200 - 2x \rightarrow x = 600$ ft.
This is a maximum since $A''(600) = -2 < 0$ and A will be concave down by the second derivative test.
Check endpoints: $A(0) = 0 = A(1200) < A(600) = 360\,000$.
The absolute maximum is 360,000 ft^2 when the rectangle is a square of side 600 ft.

Relating this to $\mathbb{R}^3$

Let's think of what is happening in $\mathbb{R}^3$ in this problem.

In the above analysis, we had an equation we wished to optimize $A = xy$. This is a surface in $\mathbb{R}^3$. Here is a sketch:

```
surface[x_, y_] = x*y
xf = 1400
yf = 1400
surfaceplot =
  Plot3D[surface[x, y], {x, 0, xf}, {y, 0, yf}, PlotPoints -> 30, ColorFunction -> Hue, Mesh -> False]
```
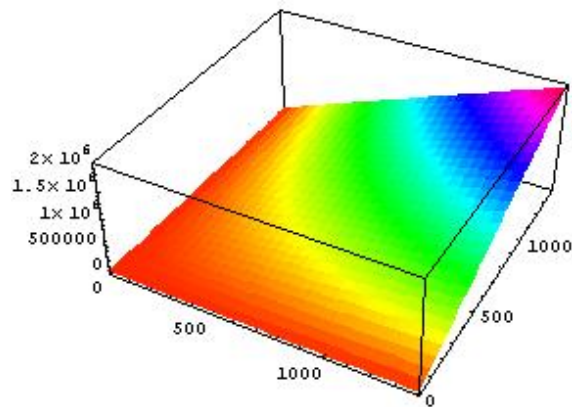
We also had a constraint equation, $P = 2x + 2y = 2400$. This means that our solution must satisfy this value for the perimeter. Using our knowledge of parametric functions, we can sketch this as a curve in $\mathbb{R}^2$:

$$
\begin{aligned}
2x + 2y &= 2400 \\
x &= 1200 - y
\end{aligned}
$$

So a parametric representation of this function in $\mathbb{R}^2$ is

$$
\begin{aligned}
x &= 1200 - t \\
y &= t, \qquad t \in (-\infty, \infty)
\end{aligned}
$$

To connect the idea of the surface and the constraint, we can think of the constraint curve as a parametric surface in $\mathbb{R}^3$:

$$
\begin{aligned}
x &= 1200 - t \\
y &= t, \qquad t \in (-\infty, \infty) \\
z &= u, \qquad u \in (-\infty, \infty)
\end{aligned}
$$

There is a command `ParametricPlot3D` in *Mathematica*:

```
constraintplot = ParametricPlot3D[{1200 - t, t, u}, {t, 0, yf}, {u, 0, 400000}, AspectRatio -> 1]
```

The output from plotting the constraint surface alone is uninteresting. However, when we overlay the area surface and the constraint surface we can see what is happening topologically with our solution.

```
Show[surfaceplot, constraintplot, ViewPoint -> {-5, -5, 4}, AxesLabel -> {"x", "y", "z"}]
```
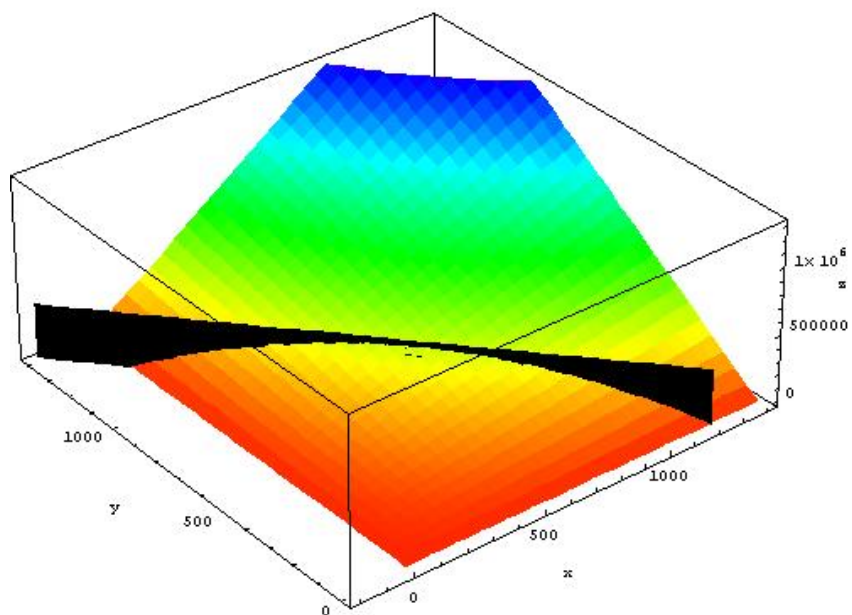
Click on the image and drag a side to make it larger if you like.

Notice how the intersection of the constraint surface and the area surface produces a space curve in $\mathbb{R}^3$. This is what we were sketching (in $\mathbb{R}^2$, of course) when we eliminated one of the variables in the area to solve the problem.

Along this space curve are all the areas that have a perimeter of 2400 ft.

The constraint surface intersects the area surface at the maximum z for $(x, y, z) = (600, 600, 360\,000)$.

This gives us a bit more topological understanding of optimization problems in $\mathbb{R}^2$!

Another Example: Minimization

A box with a square base and open top must have a volume of 32,000 cm^3. Find the dimensions of the box that minimize the amount of material used.

Diagram:
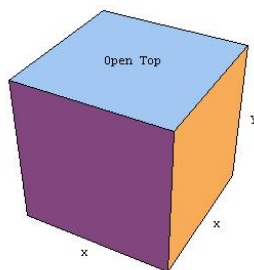
The volume is $V = x^2 y = 32000$.
The surface area is $S = 4xy + x^2$.
We want to minimize the surface are (material).
Use the volume relation to get surface are as a function of one variable:
$V = x^2 y = 32000 \longrightarrow y = 32000/x^2$.
$S(x) = 4x(\dfrac{32000}{x^2}) + x^2 = 128000\dfrac{1}{x} + x^2$.

To find the extrema, solve: $S' = -\frac{128000}{x^2} + 2x = 0$.

$$
\begin{aligned}
-\frac{128000}{x^2} + 2x &= 0 \\
\frac{128000}{x^2} &= 2x \\
128000 &= 2x^3 \\
64000 &= x^3 \\
x &= (6400)^{1/3} = 40
\end{aligned}
$$

Therefore, a box with no top and square base of 40 cm and height $y = 32000/1600 = 20$ cm will minimize the surface area and have a volume of 32000 cm.

Relating this to $\mathbb{R}^3$

In the above analysis, we had an equation we wished to optimize, the surface area $S = 4xy + x^2$. This is a surface in $\mathbb{R}^3$. Here is a sketch:
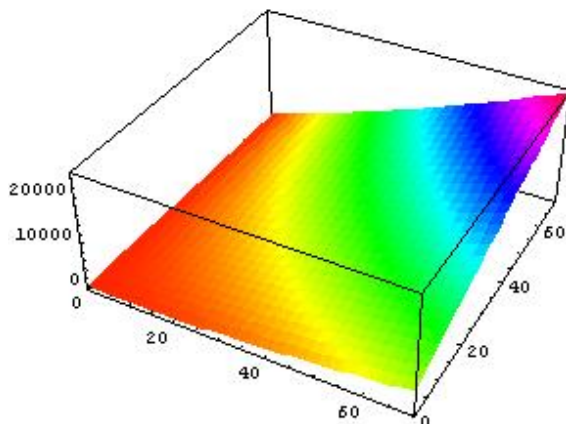
```
surface[x_, y_] = 4x*y + x^2
xf = 70
yf = 70
surfaceplot =
  Plot3D[surface[x, y], {x, 0, xf}, {y, 0, yf}, PlotPoints -> 30, ColorFunction -> Hue, Mesh -> False]
```

We also had a constraint equation, $V = x^2y = 32000$. This means that our solution must satisfy this value for the perimeter. Using our knowledge of parametric functions, we can sketch this as a CURVE in $\mathbb{R}^2$:

$$
\begin{aligned}
x^2y &= 32000 \\
x &= \frac{80\sqrt{5}}{y}, \qquad\qquad \text{exclude negative values of } x
\end{aligned}
$$

So a parametric representation of this function in $\mathbb{R}^2$ is

$$
\begin{aligned}
x &= \frac{80\sqrt{5}}{y} \\
y &= t, \qquad t \in (-\infty, \infty)
\end{aligned}
$$

To connect the surface and the constraint in $\mathbb{R}^3$, we can think of the constraint curve as a parametric surface in $\mathbb{R}^3$:

$$
\begin{aligned}
x &= \frac{80\sqrt{5}}{y} \\
y &= t, \qquad t \in (-\infty, \infty) \\
z &= u, \qquad u \in (-\infty, \infty)
\end{aligned}
$$

Sketching,

```
constraintplot = ParametricPlot3D[{80*Sqrt[5]/Sqrt[t], t, u}, {t, 0, yf}, {u, 0, 7000}, AspectRatio -> 1]
```

The output from plotting the constraint surface alone is uninteresting. However, when we overlay the surface area surface and the constraint surface we can see what is happening topologically with our solution.

```
Show[surfaceplot, constraintplot, ViewPoint -> {-5, -5, 4}, AxesLabel -> {"x", "y", "z"}]
```
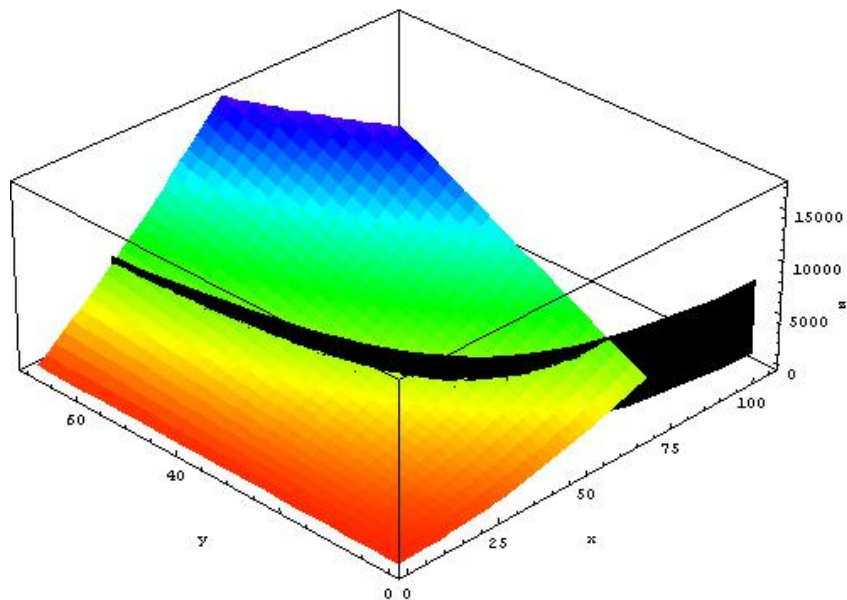
Notice how the intersection of the constraint surface and the surface area surface produces a space curve in $\mathbb{R}^3$. This is what we were sketching (in $\mathbb{R}^2$, of course) when we eliminated one of the variables in the area to solve the problem.

Along this space curve are all the surface areas that produce a box with a volume of 32000 cm.

The constraint surface intersects the surface area surface at the minimum z for $(x, y, z) = (40, 20, 4800)$.