Note It is generally alright to reuse letters (y, f, u, w, etc.) in your solution to a problem as long as they weren't part of the original question. For example, in Example 6, I reuse y, f, u (they stand for different things at different times in my solution) since they are quantities *I* introduced in my solution. However, in Example 6, y was part of the original question, so I do not reuse that variable in my solution. The exception to this is that you can reuse y if your answer is an equation of a tangent line!

Note Sometimes you have to start using logarithmic differentiation, sometimes you start using a chain rule, etc. It just depends what you are working on how you should start. Use good notation to help you keep track of parts of your solution, especially when using a chain rule. If you encounter $\frac{d}{dx}[f(x)^{g(x)}]$, break out $u = f(x)^{g(x)}$ and use logarithmic differentiation to determine $\frac{du}{dx} = \frac{d}{dx}[f(x)^{g(x)}]$ and sub this back in.

Note on Trigonometric Functions When dealing with trig functions, we must work in radians rather than degrees, since the integral formulas for the trig functions we derived were all based on the angle being measured in radians. Recall that to switch between the two measures, we can think that a circle is swept out when the angle goes through 360 degrees, or 2π radians: 2π radians = 360 degrees .

You should memorize or be able to work out <u>at least</u> the following:

$$\frac{d}{dx}[cf(x)] = \qquad \frac{d}{dx}[x^n] = \qquad \frac{d}{dx}[\csc x] =$$

$$\frac{d}{dx}[f(x)+g(x)] = \qquad \frac{d}{dx}[\ln|x|] = \qquad \frac{d}{dx}[\sec x] =$$

$$\frac{d}{dx}[f(g(x))] = \qquad \frac{d}{dx}[e^x] = \qquad \frac{d}{dx}[\cot x] =$$

$$\frac{d}{dx}[f(x)g(x)] = \qquad \frac{d}{dx}[\sin x] = \qquad \frac{d}{dx}[\arcsin x] =$$

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \qquad \frac{d}{dx}[\cos x] = \qquad \frac{d}{dx}[\arccos x] =$$

$$\frac{d}{dx}[f(g(x))] = \qquad \frac{d}{dx}[\tan x] = \qquad \frac{d}{dx}[\arctan x] =$$

1. If $y = \arctan(x^x)$, find y'.

2. Find the equation of the tangent line to the curve
$y = x + \cos x$ at $x = \frac{\pi}{3}$.

3. Find the derivative of $g(t) = \dfrac{\sin^2 t}{\cos t}$.

4. Use the chain rule to prove that the derivative of an even function is an odd function.

5. If n is a positive integer, prove that
$\frac{d}{dx}[\sin^n x \cos(nx)] = n \sin^{n-1} x \cos((n+1)x)$.

6. $y = e^{-5x} \cos 3x$, find $y'(x)$.

7. $f(t) = \dfrac{1}{(t^2 - 2t - 5)^4}$, find $f'(t)$.

8. For what values of x does the curve $y = x + \cos 2x$ have horizontal tangents?

9. $y = \sin(\ln x)$. Find y'.

10. The displacement of a particle on a vibrating string is given by $s = A\cos(\omega t + \delta)$. Find the velocity of the particle at time t. When is the velocity zero?

11. Find the x-coordinates in $(-\pi, \pi)$ for which the curve $y = \sin(2x) - 2\sin x$ has a horizontal tangent line. *Solution requires use of Mathematica to solve an equation, or you can use trig identities to solve by hand.*

12. Given $f(x) = \dfrac{\sqrt{x^2+1}}{\sec x \sin x + e^{2x}}$, find $f'(x)$.

13. Given $f(x) = \cos(\cos(\cos(\cos x)))$, find $f'(x)$.

14. Where does the normal line to the ellipse $x^2 - xy + y^2 = 3$ at the point $(-1, 1)$ intersect the ellipse a second time?

15. Suppose f is a one-to-one differentiable function and its inverse f^{-1} is also differentiable. Use implicit differentiation to show

$$\frac{d}{dx}[f^{-1}(x)] = \frac{1}{f'(f^{-1}(x))}$$

16. If $\ln(xy) = \tan^{-1} x$, find
$y' = \dfrac{dy}{dx}$ and
$y'' = \dfrac{d^2y}{dx^2} = \dfrac{d}{dx}\left[\dfrac{dy}{dx}\right]$.

17. Find y' given $y = x^{\cos x}$.

18. Find the equation of the tangent line to the curve $y = f(x) = x^2\cos^3 x + 12$ for any one point on the curve where the tangent line is horizontal.

Solutions

1. If $y = \arctan(x^x)$, find y'.

Solution:

$$\begin{aligned} y' = \frac{dy}{dx} &= \frac{d}{dx}[y] = \frac{d}{dx}[\arctan(x^x)] \\ &= \frac{d}{dx}[\arctan(u)], \quad u = x^x \\ &= \frac{d}{du}[\arctan(u)] \cdot \frac{du}{dx} \quad \text{chain rule} \end{aligned}$$

So now we need two derivatives. If you need to, you can work out the derivative of $\arctan u$ with respect to u using logarithmic differentiation, or if you have it memorized just write it down.

$$\frac{d}{du}[\arctan u] = \frac{1}{1+u^2}$$

The other derivative du/dx requires logarithmic differentiation, so start by taking a logarithm of $u = x^x$:

$$\ln[u = x^x]$$
$$\ln u = x \ln x$$
$$\frac{d}{dx}[\ln u = x \ln x]$$
$$\frac{d}{du}[\ln u]\frac{du}{dx} = \frac{d}{dx}[x \ln x]$$
$$\frac{1}{u}\frac{du}{dx} = x\frac{d}{dx}[\ln x] + \ln x \frac{d}{dx}[x]$$
$$\frac{du}{dx} = u\left(x\frac{1}{x} + \ln x\right)$$
$$\frac{du}{dx} = x^x(1 + \ln x)$$

Put it all back together, and substitute back $u = x^x$:

$$y' = \frac{1}{1+u^2} \cdot x^x(1+\ln x) = \frac{x^x(1+\ln x)}{1+x^{2x}}$$

2. Find the equation of the tangent line to the curve $y = x + \cos x$ at $x = \frac{\pi}{3}$.

Solution Statements:
The slope of the tangent line is the derivative of the function.
The point we are interested in is $(x_0, y_0) = (\frac{\pi}{3}, f(\frac{\pi}{3})) = (\frac{\pi}{3}, \frac{\pi}{3} + \frac{1}{2})$, which has $x = \frac{\pi}{3}$.
We want to find the derivative $f'(\frac{\pi}{3}) = m$.
We need to define $f(x) = x + \cos x$.
We want the equation of the tangent line, so our answer will look like $y - y_0 = m(x - x_0)$.
Our answer will look like $y - \frac{\pi}{3} - \frac{1}{2} = f'(\frac{\pi}{3})(x - \frac{\pi}{3})$.

$$\begin{aligned} f'(x) &= \frac{d}{dx}[x + \cos x] = 1 - \sin[x] \\ f'(\frac{\pi}{3}) &= 1 - \sin\frac{\pi}{3} = 1 - \frac{\sqrt{3}}{2} \end{aligned}$$

The equation of the tangent line to the curve at $x = \frac{\pi}{3}$ is

$$\begin{aligned} y - \frac{\pi}{3} - \frac{1}{2} &= (1 - \frac{\sqrt{3}}{2})(x - \frac{\pi}{3}) \\ y &= (1 - \frac{\sqrt{3}}{2})(x - \frac{\pi}{3}) + \frac{\pi}{3} + \frac{1}{2} \end{aligned}$$

3. Find the derivative of $g(t) = \dfrac{\sin^2 t}{\cos t}$.

Solution

$$\begin{aligned} g'(t) &= \frac{d}{dt}\left[\frac{\sin^2 t}{\cos t}\right] \\ &= \frac{\cos t \frac{d}{dt}[\sin^2 t] - \sin^2 t \frac{d}{dt}[\cos t]}{\cos^2 t} \quad \text{quotient rule} \end{aligned}$$

The derivative of $\sin^2 t$ will require the chain rule.

$$y = \sin^2 t \text{ decomposition: } \begin{aligned} y &= u^2 \\ u &= \sin t \end{aligned}$$

$$\begin{aligned} \frac{d}{dt}[\sin^2 t] &= \frac{dy}{dt} \\ &= \frac{dy}{du} \cdot \frac{du}{dt} \quad \text{chain rule} \\ &= (2u) \cdot (\cos t) \\ &= 2 \sin t \cos t \end{aligned}$$

$$\begin{aligned} g'(t) &= \frac{\cos t (2 \sin t \cos t) - \sin^2 t[-\sin t]}{\cos^2 t} \\ &= \frac{2 \sin t \cos^2 t + \sin^3 t}{\cos^2 t} \\ &= 2 \sin t + \sin t \tan^2 t \end{aligned}$$

4. Use the chain rule to prove that the derivative of an even function is an odd function.

Solution An even function will satisfy the equation:

$$\begin{aligned} f(x) &= f(-x) \quad \text{differentiate this equation} \\ \frac{d}{dx} f(x) &= \frac{d}{dx} f(-x) \\ f'(x) &= \frac{d}{dx} f(u), \quad u = -x \\ &= \frac{d}{du} f(u) \cdot \frac{du}{dx} \quad \text{chain rule} \\ &= f'(u) \cdot (-1) \\ &= -f'(-x) \end{aligned}$$

since $f'(x) = -f'(-x)$, $f'(x)$ is odd! Much easier than our previous proof of this result using the definition of derivative (see Homework Section 2.9).

5. If n is a positive integer, prove that $\frac{d}{dx}[\sin^n x\cos(nx)] = n\sin^{n-1} x\cos((n+1)x)$.

Solution

$$\frac{d}{dx}[\sin^n x\cos(nx)] = \frac{d}{dx}[\sin^n x]\cos(nx) + \sin^n x\frac{d}{dx}[\cos(nx)] \quad \text{product rule}$$

We need to use the chain rule to do the two derivatives. Let's do 'em!

$y = \sin^n x$ decomposition:
$$\begin{aligned} y &= u^n \\ u &= \sin x \end{aligned}$$

$$\begin{aligned} \frac{d}{dx}[\sin^n x] &= \frac{dy}{du}\cdot\frac{du}{dx} \quad \text{chain rule} \\ &= nu^{n-1}\cdot\cos x \\ &= n\sin^{n-1} x\cos x \end{aligned}$$

$y = \cos(nx)$ decomposition:
$$\begin{aligned} y &= \cos u \\ u &= nx \end{aligned}$$

$$\begin{aligned} \frac{d}{dx}[\cos(nx)] &= \frac{dy}{du}\cdot\frac{du}{dx} \quad \text{chain rule} \\ &= (-\sin u)\cdot n \\ &= -n\sin(nx) \end{aligned}$$

Now we substitute back:

$$\begin{aligned} \frac{d}{dx}[\sin^n x\cos(nx)] &= \frac{d}{dx}[\sin^n x]\cos(nx) + \sin^n x\frac{d}{dx}[\cos(nx)] \quad \text{product rule} \\ &= n\sin^{n-1} x\cos x\cos(nx) + \sin^n x(-n\sin(nx)) \\ &= n\left(\sin^{n-1} x\cos x\cos(nx) - \sin^n x\sin(nx)\right) \\ &= n\sin^{n-1} x\left(\cos x\cos(nx) - \sin x\sin(nx)\right) \end{aligned}$$

Use the trig identity $\cos a\cos b - \sin a\sin b = \cos(a+b)$ to rewrite $\cos x\cos(nx) - \sin x\sin(nx) = \cos[(n+1)x]$ and we arrive at the answer,

$$\frac{d}{dx}[\sin^n x\cos(nx)] = n\sin^{n-1} x\cos[(n+1)x]$$

6. $y = e^{-5x}\cos 3x$, find $y'(x)$.

Solution

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}[e^{-5x}\cos 3x] \\ &= \frac{d}{dx}[e^{-5x}]\cos 3x + e^{-5x}\frac{d}{dx}[\cos 3x] \quad \text{product rule}\end{aligned}$$

We must use the chain rule to do the two derivatives.

$$f = e^{-5x} \text{ decomposition: } \begin{aligned} f &= e^u \\ u &= -5x \end{aligned}$$

$$\begin{aligned}\frac{df}{dx} &= \frac{df}{du}\cdot\frac{du}{dx} \quad \text{chain rule} \\ &= e^u \cdot (-5) \\ &= -5e^{-5x}\end{aligned}$$

$$f = \cos 3x \text{ decomposition: } \begin{aligned} f &= \cos u \\ u &= 3x \end{aligned}$$

$$\begin{aligned}\frac{df}{dx} &= \frac{df}{du}\cdot\frac{du}{dx} \quad \text{chain rule} \\ &= -\sin u \cdot (3) \\ &= -3\sin 3x\end{aligned}$$

Now we can substitute back:

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}[e^{-5x}\cos 3x] \\ &= \frac{d}{dx}[e^{-5x}]\cos 3x + e^{-5x}\frac{d}{dx}[\cos 3x] \\ &= -5e^{-5x}\cos 3x + e^{-5x}(-3\sin 3x) \\ &= -e^{-5x}(5\cos 3x + 3\sin 3x)\end{aligned}$$

7. $f(t) = \dfrac{1}{(t^2-2t-5)^4}$, find $f'(t)$.

Solution

$$\begin{aligned} f'(t) &= \frac{(t^2-2t-5)^4 \frac{d}{dt}[1] - (1)\frac{d}{dt}\left[(t^2-2t-5)^4\right]}{(t^2-2t-5)^8} && \text{quotient rule} \\ &= -\frac{\frac{d}{dt}\left[(t^2-2t-5)^4\right]}{(t^2-2t-5)^8} && \text{constant rule} \end{aligned}$$

We need to use the chain rule to do this derivative:

$$y = (t^2-2t-5)^4 \text{ decomposition: } \begin{aligned} y &= u^4 \\ u &= t^2-2t-5 \end{aligned}$$

$$\begin{aligned} \frac{dy}{dt} &= \frac{dy}{du}\cdot\frac{du}{dt} && \text{chain rule} \\ &= (4u^3)\cdot(2t-2) \\ &= 4(2t-2)(t^2-2t-5)^3 \end{aligned}$$

Now we can substitute back:

$$\begin{aligned} f'(t) &= -\frac{\frac{d}{dt}\left[(t^2-2t-5)^4\right]}{(t^2-2t-5)^8} \\ &= -\frac{4(2t-2)(t^2-2t-5)^3}{(t^2-2t-5)^8} \\ &= -\frac{4(2t-2)}{(t^2-2t-5)^5} \end{aligned}$$

Alternate solution Rewrite the function as $f(t) = (t^2-2t-5)^{-4}$. $f'(t) = \dfrac{d}{dt}\left[(t^2-2t-5)^{-4}\right]$.
We need to use the chain rule to do this derivative:

$$f = (t^2-2t-5)^{-4} \text{ decomposition: } \begin{aligned} f &= u^{-4} \\ u &= t^2-2t-5 \end{aligned}$$

$$\begin{aligned} \frac{df}{dt} &= \frac{df}{du}\cdot\frac{du}{dt} && \text{chain rule} \\ &= (-4u^{-5})\cdot(2t-2) \\ &= -4(2t-2)(t^2-2t-5)^{-5} \\ f'(t) = \frac{df}{dt} &= -\frac{4(2t-2)}{(t^2-2t-5)^5} \end{aligned}$$

8. For what values of x does the curve $y = x + \cos 2x$ have horizontal tangents?

Solution Statements:
The slope of the tangent line is the derivative of the function.
We want the values of x, let's call them $x = a$, for which the tangent is horizontal.
We need to solve $f'(a) = 0$ for the number a.
We need to define $f(x) = x + \cos 2x$.
Our answer will be the numbers a.

$$\begin{aligned}
f'(x) &= \frac{d}{dx}[x + \cos 2x] \\
&= 1 + \frac{d}{dx}[\cos 2x] \\
&\quad \text{decompose: } u = 2x, \\
&= 1 + \frac{d}{dx}[\cos u] \\
&= 1 + \frac{d}{du}[\cos u] \cdot \frac{du}{dx} \quad \text{(chain rule)} \\
&= 1 + (-\sin u) \cdot (2) \\
&= 1 - 2\sin 2x \\
f'(a) = 0 \quad &\longrightarrow \quad 1 - 2\sin 2a = 0 \\
&\quad \sin 2a = \frac{1}{2}
\end{aligned}$$

This can be solved by noting that $\sin(\pi/6) = 1/2$. This is one of our special angles.
There are other solutions since sine is periodic with period 2π, $\sin(\pi/6 + 2n\pi) = 1/2$, n is an integer.
Therefore, $2a = \dfrac{\pi}{6} + 2n\pi$, n is an integer.
Therefore, $a = \dfrac{\pi}{12} + n\pi$, n is an integer.

Since it is also true that $\sin(5\pi/6) = 1/2$, we get more solutions:

$$\begin{aligned}
2a &= \frac{5\pi}{6} + 2n\pi, n \text{ is an integer} \\
a &= \frac{5\pi}{12} + n\pi.
\end{aligned}$$

9. $y = \sin(\ln x)$. Find y'.

Solution Here, y is a function of x, so $y\prime = dy/dx$.

$$y = \sin(e^x) \text{ decomposition: } \begin{aligned} y &= \sin u \\ u &= \ln x \end{aligned}$$

$$\begin{aligned}
\frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} \quad \text{chain rule} \\
&= (\cos u) \cdot \left(\frac{1}{x}\right) \\
&= \frac{\cos(\ln x)}{x}
\end{aligned}$$

10. The displacement of a particle on a vibrating string is given by $s = A\cos(\omega t + \delta)$. Find the velocity of the particle at time t. When is the velocity zero?

Solution Statements:
The velocity is the derivative of the position function.
The position function is given by $s = f(t)$.
The velocity will be $v(t) = f'(t)$.
We will need to use the chain rule to calculate the derivative.
Once we have the velocity, we can determine for what time it is zero by solving $v(t) = 0$ for t.

$$f(t) = A\cos(\omega t + \delta) \text{ decomposition: } \begin{aligned} f &= A\cos h \\ h &= \omega t + \delta \end{aligned}$$

$$\begin{aligned} v(t) = f'(t) &= \frac{df}{dt} \\ &= \frac{df}{dh} \cdot \frac{dh}{dt} \quad \text{chain rule} \\ &= \frac{d}{dh}[A\cos h]\frac{d}{dt}[\omega t + \delta] \\ &= A(-\sin h)(\omega) \\ &= -A\omega\sin(\omega t + \delta) \end{aligned}$$

The velocity is zero when

$$-A\omega\sin(\omega t + \delta) = 0$$

which occurs when $\omega t + \delta = n\pi$, n an integer, so $t = (n\pi - \delta)/\omega$.

11. Find the x-coordinates in $(-\pi, \pi)$ for which the curve $y = \sin(2x) - 2\sin x$ has a horizontal tangent line. *Solution requires use of Mathematica to solve an equation.*

Solution Statements:
Slope of the tangent line is the derivative of the function.
Our function will be $f(x) = \sin(2x) - 2\sin x$.
We will have to use the chain rule to determine the derivative ($\sin 2x$).
If the tangent line is horizontal, then the slope is zero.
We want to find all the points a which satisfy $f'(a) = 0$.

$$\begin{aligned} f'(x) &= \frac{d}{dx}[\sin(2x) - 2\sin x] \\ &= \frac{d}{dx}[\sin(2x)] - 2\frac{d}{dx}[\sin x] \\ \frac{d}{dx}[\sin(2x)] &= \frac{d}{dx}[\sin u], \;\; u = 2x \\ &= \frac{d}{du}[\sin u]\frac{du}{dx} \quad \text{chain rule} \\ &= \cos u \cdot (2) \\ &= 2\cos(2x) \\ f'(x) &= 2\cos(2x) - 2\cos x \end{aligned}$$

Use *Mathematica* to solve for a in $f'(a) = 0$: `Solve[2 Cos[2*a] - 2 Cos[a] == 0, x]`

And we find that $a = 0, -2\pi/3, 2\pi/3$.

12. Given $f(x) = \dfrac{\sqrt{x^2+1}}{\sec x \sin x + e^{2x}}$, find $f'(x)$.

Solution

$$\begin{aligned} f'(x) &= \frac{d}{dx}\left[\frac{\sqrt{x^2+1}}{\sec x \sin x + e^{2x}}\right] \\ &= \frac{(\sec x \sin x + e^{2x})\frac{d}{dx}[\sqrt{x^2+1}] - (\sqrt{x^2+1})\frac{d}{dx}[\sec x \sin x + e^{2x}]}{(\sec x \sin x + e^{2x})^2} \quad \text{(quotient rule)} \qquad (1) \end{aligned}$$

Let's pause to work out the two derivatives as an aside.

$$\begin{aligned} \frac{d}{dx}[\sqrt{x^2+1}] &= \frac{d}{dx}[\sqrt{u}], \quad u = x^2+1 \\ &= \frac{d}{du}[\sqrt{u}]\cdot\frac{du}{dx}, \quad \text{(chain rule)} \\ &= \frac{1}{2}u^{1/2-1}\cdot(2x), \\ &= \frac{1}{2\sqrt{u}}\cdot(2x), \\ &= \frac{x}{\sqrt{x^2+1}}. \qquad (2) \\ \frac{d}{dx}[\sec x \sin x + e^{2x}] &= \frac{d}{dx}[\sec x \sin x] + \frac{d}{dx}[e^{2x}] \quad \text{(sum rule)} \\ &= \frac{d}{dx}[\sec x]\sin x + \sec x\frac{d}{dx}[\sin x] + \frac{d}{dx}[e^u], \quad u = 2x \\ &\quad \text{(first term: product rule; second term: set up for chain rule)} \\ &= [\sec x \tan x]\sin x + \sec x[\cos x] + \frac{d}{du}[e^u]\cdot\frac{du}{dx}, \quad \text{(chain rule)} \\ &= [\frac{1}{\cos x}\cdot\frac{\sin x}{\cos x}]\sin x + \sec x[\frac{1}{\sec x}] + [e^u]\cdot(2), \quad \text{(simplify)} \\ &= \tan^2 x + 1 + 2e^{2x}. \qquad (3) \end{aligned}$$

Now, we can substitute Equations (2) and (3) into Equation (1).

$$\begin{aligned} f'(x) &= \frac{(\sec x \sin x + e^{2x})\frac{d}{dx}[\sqrt{x^2+1}] - (\sqrt{x^2+1})\frac{d}{dx}[\sec x \sin x + e^{2x}]}{(\sec x \sin x + e^{2x})^2} \\ &= \frac{(\sec x \sin x + e^{2x})\left(\frac{x}{\sqrt{x^2+1}}\right) - (\sqrt{x^2+1})[\tan^2 x + 1 + 2e^{2x}]}{(\sec x \sin x + e^{2x})^2} \end{aligned}$$

If we need to, we can simplify this. However, we don't have to if all we wanted was the derivative.

If we want to compare with what *Mathematica* gives us, we need to simplify a bit. Use the following: $\tan^2 x + 1 = \sec^2 x$, and $\sec x \sin x = \tan x$:

$$\begin{aligned} f'(x) &= \frac{(\tan x + e^{2x})\left(\frac{x}{\sqrt{x^2+1}}\right) - (\sqrt{x^2+1})[\sec^2 x + 2e^{2x}]}{(\tan x + e^{2x})^2} \\ &= \frac{(\tan x + e^{2x})\left(\frac{x}{\sqrt{x^2+1}}\right)}{(\tan x + e^{2x})^2} - \frac{(\sqrt{x^2+1})[\sec^2 x + 2e^{2x}]}{(\tan x + e^{2x})^2} \\ &= \frac{x}{\sqrt{x^2+1}(\tan x + e^{2x})} - \frac{(\sqrt{x^2+1})(\sec^2 x + 2e^{2x})}{(\tan x + e^{2x})^2} \end{aligned}$$

13. Given $f(x) = \cos(\cos(\cos(\cos x)))$, find $f'(x)$.

Solution This is a test of our chain rule abilities. Let's decompose!

$s = \cos w$, $w = \cos v$, $v = \cos u$, $u = \cos x$.

Check we did the decomposition correctly:

$$\begin{aligned}
(s \circ w \circ v \circ u)(x) &= s\left(w\Big(v\big(u(x)\big)\Big)\right) \\
&= s\left(w\Big(v\big(\cos x\big)\Big)\right) \\
&= s\left(w\Big(\cos\big(\cos x\big)\Big)\right) \\
&= s\left(\cos(\cos(\cos x))\right) \\
&= \cos(\cos(\cos(\cos x))) \\
&= f(x)
\end{aligned}$$

Therefore, we can use the chain rule as follows.

$$\begin{aligned}
f'(x) &= \frac{ds}{dw} \cdot \frac{dw}{dv} \cdot \frac{dv}{du} \cdot \frac{du}{dx} \\
&= \frac{d}{dw}[\cos w] \cdot \frac{d}{dv}[\cos v] \cdot \frac{d}{du}[\cos u] \cdot \frac{d}{dx}[\cos x] \\
&= [-\sin w] \cdot [-\sin v] \cdot [-\sin u] \cdot [-\sin x] \\
&= [\sin(\cos v)] \cdot [\sin(\cos u)] \cdot [\sin \cos x] \cdot [\sin x] \\
&= [\sin(\cos(\cos u))] \cdot [\sin(\cos(\cos x))] \cdot [\sin(\cos x)] \cdot [\sin x] \\
&= [\sin(\cos(\cos(\cos x)))] \cdot [\sin(\cos(\cos x))] \cdot [\sin(\cos x)] \cdot [\sin x]
\end{aligned}$$

14. Where does the normal line to the ellipse $x^2 - xy + y^2 = 3$ at the point $(-1, 1)$ intersect the ellipse a second time?

Solution To find the normal line, we will need to first find the slope of the tangent line. That means we need the derivative, and since this is an implicit function, we want to implicitly differentiate.

$$\begin{aligned}
&\frac{d}{dx}[x^2 - xy + y^2] = \frac{d}{dx}[3] \\
&\frac{d}{dx}[x^2] - \frac{d}{dx}[xy] + \frac{d}{dx}[y^2] = 0 \\
&2x - \left(x\frac{d}{dx}[y] + y\frac{d}{dx}[x]\right) + \frac{d}{dy}[y^2]\frac{dy}{dx} = 0 \\
&2x - \left(x\frac{dy}{dx} + y\right) + 2y\frac{dy}{dx} = 0 \\
&2x - x\frac{dy}{dx} - y + 2y\frac{dy}{dx} = 0 \\
&\frac{dy}{dx}\Big(-x + 2y\Big) = y - 2x \\
&\frac{dy}{dx} = \frac{y - 2x}{2y - x}
\end{aligned}$$

The slope of the tangent line at $(-1, 1)$ is therefore $\left.\frac{dy}{dx}\right|_{(-1,1)} = \frac{1 - 2(-1)}{2(1) - (-1)} = 1.$

The slope of the normal line at $(-1, 1)$ is therefore $m = -1$ (perpendicular lines have slopes that are negative reciprocals).

The equation of the normal line is

$$\begin{aligned}
y - y_0 &= m(x - x_0) \\
y - 1 &= -1(x + 1) \\
y &= -x
\end{aligned}$$

Two find where this intersects the ellipse, solve the system of equations

$$\begin{aligned} &x^2 - xy + y^2 = 3 \\ &y = -x \end{aligned}$$

for (x, y). Take the second equation and substitute it into the first:

$$\begin{aligned} &x^2 - x(-x) + (-x)^2 = 3 \\ &x^2 + x^2 + x^2 = 3 \\ &x^2 = 1 \\ &x = +1 \text{ or } -1 \end{aligned}$$

If $x = -1$, we get $y = -x = 1$, so the point $(-1, 1)$.

If $x = +1$, we get $y = -x = -1$, so the point $(1, -1)$.

You can check this by graphing in *Mathematica*:

```
p1 = ContourPlot[x^2 - x y + y^2 == 3, {x, -5, 5}, {y, -5, 5}];
p2 = Plot[-x, {x, -5, 5}];
Show[p1, p2, AspectRatio -> 1]
```

15. Suppose f is a one-to-one differentiable function and its inverse f^{-1} is also differentiable. Use implicit differentiation to show

$$\frac{d}{dx}[f^{-1}(x)] = \frac{1}{f'(f^{-1}(x))}$$

Solution Start by writing down one of the cancelation equations for inverse functions. Then implicitly differentiate.

$$\begin{aligned} f(f^{-1}(x)) &= x \\ \frac{d}{dx}[f(f^{-1}(x))] &= \frac{d}{dx}[x] \\ \frac{d}{dx}[f(u)] &= 1 \text{ let } u = f^{-1}(x) \\ \frac{d}{du}[f(u)]\frac{du}{dx} &= 1 \text{ use the chain rule} \\ f'(u)\frac{du}{dx} &= 1 \\ \frac{du}{dx} &= \frac{1}{f'(u)} \text{ put } u \text{ back in} \\ \frac{d}{dx}[f^{-1}(x)] &= \frac{1}{f'(f^{-1}(x))} \end{aligned}$$

16. If $\ln(xy) = \tan^{-1} x$, find $y' = \dfrac{dy}{dx}$ and $y'' = \dfrac{d^2y}{dx^2} = \dfrac{d}{dx}\left[\dfrac{dy}{dx}\right]$.

Solution Since the function is implicitly defined, we must do this with implicit differentiation.

$$\begin{aligned}
\frac{d}{dx}[\ln(xy)] &= \frac{d}{dx}[\tan^{-1} x] \\
\frac{\frac{d}{dx}[xy]}{xy} &= \frac{1}{1+x^2} \\
\frac{y\frac{d}{dx}[x] + x\frac{d}{dx}[y]}{xy} &= \frac{1}{1+x^2} \\
\frac{y + x\frac{dy}{dx}}{xy} &= \frac{1}{1+x^2} \\
\frac{dy}{dx} &= \frac{y}{1+x^2} - \frac{y}{x} \\
\frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{d}{dx}\right] &= \frac{d}{dx}\left[\frac{y}{1+x^2} - \frac{y}{x}\right] \\
&= \frac{d}{dx}\left[\frac{y}{1+x^2}\right] - \frac{d}{dx}\left[\frac{y}{x}\right] \\
&= \frac{(1+x^2)\frac{d}{dx}[y] - y\frac{d}{dx}[1+x^2]}{(1+x^2)^2} - \frac{x\frac{d}{dx}[y] - y\frac{d}{dx}[x]}{x^2} \\
&= \frac{(1+x^2)\frac{dy}{dx} - y[2x]}{(1+x^2)^2} - \frac{x\frac{dy}{dx} - y}{x^2} \\
&= \frac{(1+x^2)\left(\frac{y}{1+x^2} - \frac{y}{x}\right) - 2xy}{(1+x^2)^2} - \frac{x\left(\frac{y}{1+x^2} - \frac{y}{x}\right) - y}{x^2}
\end{aligned}$$

You could also simplify dy/dx and instead write

$$\begin{aligned}
\frac{dy}{dx} &= y\left(\frac{1}{1+x^2} - \frac{1}{x}\right) \\
\frac{d^2y}{dx^2} &= \frac{d}{dx}\left[y\left(\frac{1}{1+x^2} - \frac{1}{x}\right)\right] \\
&= \frac{d}{dx}[y]\left(\frac{1}{1+x^2} - \frac{1}{x}\right) + y\frac{d}{dx}\left[\left(\frac{1}{1+x^2} - \frac{1}{x}\right)\right] \\
&= \frac{dy}{dx}\left(\frac{1}{1+x^2} - \frac{1}{x}\right) + y\left(\frac{(1+x^2)\frac{d}{dx}[1] - 1\frac{d}{dx}[1+x^2]}{(1+x^2)^2} - \frac{x\frac{d}{dx}[1] - 1\frac{d}{dx}[x]}{x^2}\right) \\
&= \frac{dy}{dx}\left(\frac{1}{1+x^2} - \frac{1}{x}\right) + y\left(\frac{-2x}{(1+x^2)^2} - \frac{-1}{x^2}\right) = y\left(\frac{1}{1+x^2} - \frac{1}{x}\right)^2 + y\left(\frac{-2x}{(1+x^2)^2} + \frac{1}{x^2}\right)
\end{aligned}$$

17. Find y' given $y = x^{\cos x}$.

Solution (logarithmic differentiation since base and exponent depend on x)

$$\begin{aligned}
\ln y &= \ln(x^{\cos x}) \\
\ln y &= \cos x \ln(x) \\
\frac{d}{dx}[\ln y] &= \frac{d}{dx}[\cos x \ln x] \\
\frac{d}{dy}[\ln y]\frac{dy}{dx} &= \cos x \frac{d}{dx}[\ln x] + \ln x \frac{d}{dx}[\cos x] \\
\frac{1}{y}\frac{dy}{dx} &= \cos x \cdot \frac{1}{x} + \ln x(-\sin x) \\
\frac{dy}{dx} &= y(\frac{\cos x}{x} - \sin x \ln x) \\
&= x^{\cos x}(\frac{\cos x}{x} - \sin x \ln x)
\end{aligned}$$

18. Find the equation of the tangent line to the curve $y = f(x) = x^2 \cos^3 x + 12$ for any one point on the curve where the tangent line is horizontal.

Solution Let's get the derivative!

$$\begin{aligned}
f'(x) &= \frac{d}{dx}[x^2 \cos^3 x + 12] = \frac{d}{dx}[x^2 \cos^3 x] + \frac{d}{dx}[12] \quad \text{sum rule} \\
&= \frac{d}{dx}[x^2] \cos^3 x + x^2 \frac{d}{dx}[\cos^3 x] + 0 \quad \text{product rule} \\
&= 2x \cos^3 x + x^2 \frac{d}{dx}[u^3], \quad u = \cos x \quad \text{power rule, set up second term for chain rule} \\
&= 2x \cos^3 x + x^2 \frac{d}{du}[u^3] \cdot \frac{du}{dx}, \quad \text{chain rule} \\
&= 2x \cos^3 x + x^2 \frac{d}{du}[u^3] \cdot \frac{d}{dx}[\cos x], \\
&= 2x \cos^3 x + x^2 \left(3u^2\right) (-\sin x), \\
&= 2x \cos^3 x + x^2 \left(3 \cos^2 x\right) (-\sin x), \\
&= 2x \cos^3 x - 3x^2 \sin x \cos^2 x,
\end{aligned}$$

To get the equation of the tangent line, we will need to find $y - y_0 = m(x - x_0)$, where (x_0, y_0) is a point on the tangent line and m is the slope of the tangent line. The slope of the tangent line will be the derivative of the function evaluated at that point, so $m = f'(x_0)$. We need to solve the equation

$$f'(x) = 2x \cos^3 x - 3x^2 \sin x \cos^2 x = 0$$

for x. One solution is $x = 0$; you could also use *Mathematica* to get all the solutions:

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f[x_] = x^2 Cos[x]^3 + 12
Solve[f'[x] == 0, x]
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Possible solutions are $x = 0, -\pi/2, \pi/2$. Let's choose $x = 0$. Therefore, $x_0 = 0$, and since $y_0 = f(0) = 12$, the equation of the tangent line is

$$\begin{aligned}
y - y_0 &= m(x - x_0) \\
y - 12 &= 0(x - 0) \\
y &= 12
\end{aligned}$$

The values of $x = \pm\pi/2$ also have tangent line $y = 12$.