4.2 The Mean Value Theorem

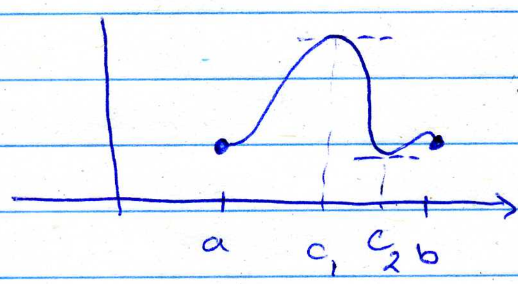
- Useful theoretical tool, especially in numerical analysis.
- proofs are in text if you are interested (proofs won't be on tests)

<u>Goal</u>: prove $\arcsin \frac{x-1}{x+1} = 2\arctan\sqrt{x} - \pi/2$.

<u>Rolle's Theorem</u>: f is a function that satisfies:

1) f is continuous on $[a,b]$
2) f is differentiable on (a,b)
3) $f(a)=f(b)$.

Then there is a number c in (a,b) such that $f'(c)=0$.



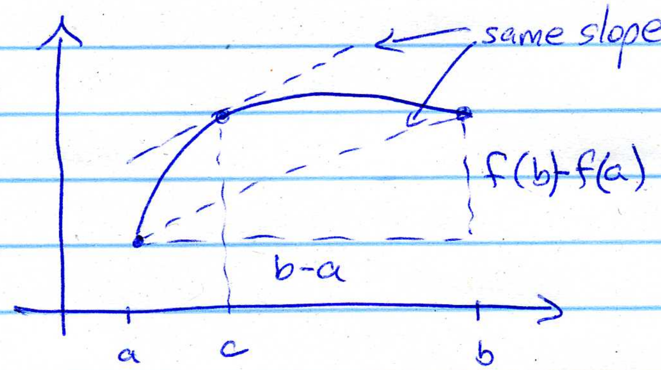
The Mean Value Theorem

f is a function that satisfies:

1) f is continuous on $[a,b]$
2) f is differentiable on (a,b)

Then there is a number $c \in (a,b)$ such that

$$f(b) - f(a) = f'(c)\,(b-a)$$



$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

Theorem If $f'(x) = 0$ for all $x \in (a,b)$, then f is a constant on (a,b).

Corollary If $f'(x) = g'(x)$ for all x in (a,b), then on (a,b): $f(x) = g(x) + C$ where C is a constant.

Ex] Prove: $\arcsin\left(\frac{x-1}{x+1}\right) = 2\arctan\sqrt{x} - \pi/2$.

$f(x) = \arcsin\left(\frac{x-1}{x+1}\right)$ $\quad g(x) = 2\arctan\sqrt{x}$ $\quad c = -\pi/2$.

$$f'(x) = \frac{d}{du}\left[\arcsin u\right] \cdot \frac{du}{dx} \qquad u = \frac{x-1}{x+1}$$

$$= \frac{1}{\sqrt{1-u^2}}\left(\frac{(x+1)(1)-(x-1)(1)}{(x+1)^2}\right)$$

$$= \frac{1}{\sqrt{1-\left(\frac{x-1}{x+1}\right)^2}}\left(\frac{2}{(x+1)^2}\right)$$

$$g'(x) = 2\frac{d}{du}\left[\arctan u\right]\frac{du}{dt} \qquad u = \sqrt{x}$$

$$= 2\,\frac{1}{1+u^2}\left(\frac{1}{2}x^{-1/2}\right)$$

$$g'(x) = \frac{1}{\sqrt{x}\,(1+x)}$$

need to simplify $f'(x)$ some more!

$$f'(x) = \frac{2}{\left(\frac{\sqrt{(x+1)^2-(x-1)^2}}{\cancel{x+1}}\right)(x+1)^2}$$

$$= \frac{2}{\sqrt{\underbrace{\left[(x+1)+(x-1)\right]}_{2x}\underbrace{\left[(x+1)-(x-1)\right]}_{2}}\;(x+1)}$$

$$= \frac{2}{\sqrt{4x}\,(x+1)} = \frac{1}{\sqrt{x}\,(x+1)}$$

So $f'(x) = g'(x)$!

$c = \arcsin\left(\frac{x-1}{x+1}\right) - 2\arctan\sqrt{x}$

evaluate at $x=1$

$\arcsin 0 = 0$

$\arctan 1 = \pi/4$

$\Rightarrow c = 0 - 2(\pi/4)$

$= -\pi/2$.

So

$f(x) = g(x) + c$

$\Rightarrow$

$\arcsin\left(\frac{x-1}{x+1}\right) = 2\arctan\sqrt{x} - \pi/2$.