1101 Calculus I 4.7 Optimization Problems

• The Method of Solution:

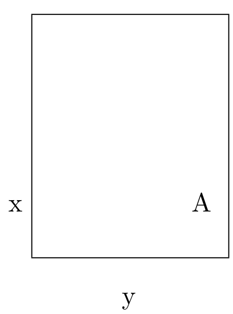
1. Understand the problem.
2. Draw a diagram.
3. Introduce notation (Q is to be maximized or minimized)
4. Find relation between quantities (Q and all others)
5. Make the relation look like $Q = f(x)$ (one variable)
6. Solve $f'(x) = 0$ for x.
7. Explain whether you have found a max or min, and if possible if it is an absolute extrema (Closed Interval Method, First Derivative Test, Second Derivative Test, argue based on the geometry of the problem)
8. Write a concluding statement.

Example A farmer has 2400 ft of fencing. What are the dimensions of the rectangular pen that produce the largest area?

• Understand the problem:

We need a rectangle. The rectangle should have maximum area for a given perimeter.

• Draw a diagram :



• Introduce notation and find relations:

The perimeter is $P = 2x + 2y$.
The area is $A = xy$. This is what we want to maximize.
We need to eliminate y from the equation for A. Use $P = 2x + 2y = 2400$, $\longrightarrow y = 1200 - x$.
Therefore, $A = xy = x(1200 - x) = 1200x - x^2$.
If $x < 0$, the area would be negative. This is unphysical.
If $x > 1200$, the area would be negative. This is unphysical.
The domain for the area is $0 \le x \le 1200$.

• Find the maximum of $A(x) = 1200x - x^2, 0 \le x \le 1200$.

$A' = 1200 - 2x$.
$A' = 0 = 1200 - 2x \to x = 600$ ft.
This is a maximum since $A''(600) = -2 < 0$ and A will be concave down by the second derivative test.
Check endpoints: $A(0) = 0 = A(1200) < A(600) = 360\,000$.
The absolute maximum is 360,000 ft^2 when the rectangle is a square of side 600 ft.

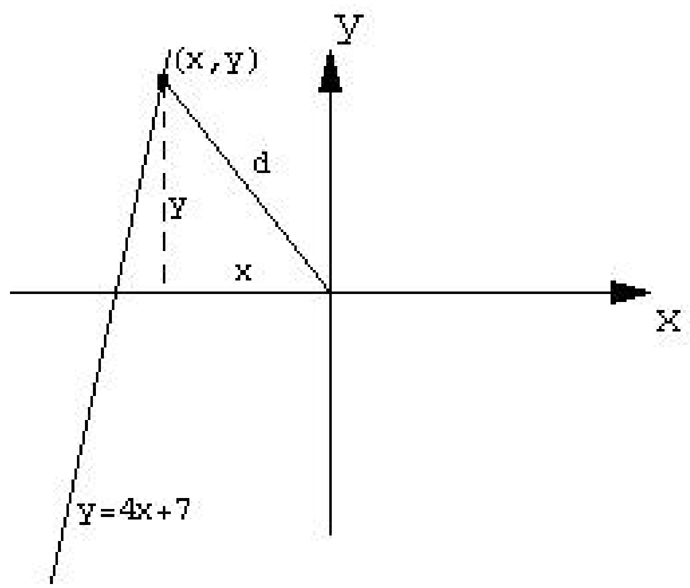
Example Find two numbers whose sum is 23 and whose product is a maximum.

For this, we don't need a graph! We want two numbers. Let:
x be the first number
y be the second number

Sum $= x + y = 23 \rightarrow y = 23 - x$.
Product $= P = xy = x(23 - x) = 23x - x^2$ needs to be maximized.
x and y could presumably be any numbers, so there is no restriction on the domain of $P(x)$.
$P'(x) = -2x + 23$.
$-2x + 23 = 0 \longrightarrow x = 23/2$.
$P''(x) = -2 < 0$ so $P(x)$ is concave down for all x, so we have found an absolute max of the product.
The two numbers are therefore $x = y = 23/2$.

Example Find the point on the line $y = 4x + 7$ that is closest to the point $(0, 0)$.

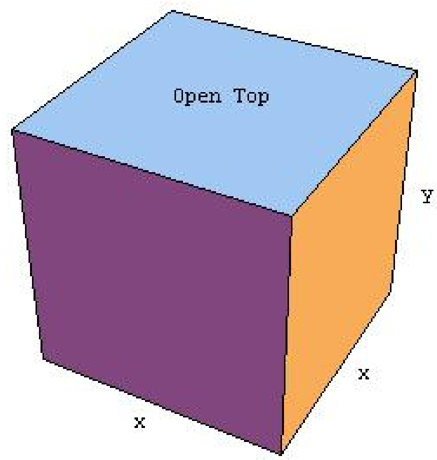
Diagram:



Write down the distance from the point $(0, 0)$ to a point (x, y):
$d^2 = (x - 0)^2 + (y - 0)^2 = x^2 + y^2$.
$f(x) = y = 4x + 7$.
We can minimize the distance squared, which will minimize the distance.
Put in the equation of the line:
$Q(x) = x^2 + (4x + 7)^2$.
$Q'(x) = 2x + 2(4x + 7)4 = 56 + 34x$.
Solve $Q'(x) = 0 = 56 + 34x \longrightarrow x = -28/17 \sim -1.64$.
Since $Q''(-28/17) = 34$, the curve is concave up and we have a minimum at $x = -28/17$.
Since the function $Q(x)$ is a quadratic, it will either have a max or a min. We have shown that it has a min, so it is an absolute min.
The point on the line closest to $(0, 0)$ is $(-28/17, f(-28/17)) = (-28/17, 7/17)$.

Example A box with a square base and open top must have a volume of 32,000 cm^3. Find the dimensions of the box that minimize the amount of material used.

Diagram:



The volume is $V = x^2 y = 32000$.
The surface area is $S = 4xy + x^2$.
We want to minimize the surface are (material).
Use the volume relation to get surface are as a function of one variable:
$V = x^2 y = 32000 \longrightarrow y = 32000/x^2$.
$S(x) = 4x(\dfrac{32000}{x^2}) + x^2 = 128000\dfrac{1}{x} + x^2$.

From the geometry of the situation, $x > 0$ since the function is infinite if $x = 0$. The surface are is infinite at $x = 0$. As x increases, the surface are will begin to decrease. there will be one point for which the surface area is minimized. After that, the surface area will increase. Therefore, the extrema we find will be a minimum.

Another way you could show that you will find a minimum is to use the second derivative test. By calculating $S''(x)$, you could show that $S''(x) \neq 0$ for $x > 0$, and then show that the function is always concave up. Therefore, we must have a minimum (and the absolute minimum at that).

To find the extrema, solve: $S' = -\frac{128000}{x^2} + 2x = 0$.

$$\begin{aligned}
-\frac{128000}{x^2} + 2x &= 0 \\
\frac{128000}{x^2} &= 2x \\
128000 &= 2x^3 \\
64000 &= x^3 \\
x &= (6400)^{1/3} = 40
\end{aligned}$$

Therefore, a box with no top and square base of 40 cm and height $y = 32000/1600 = 20$ cm will minimize the surface area and have a volume of 32000 cm.