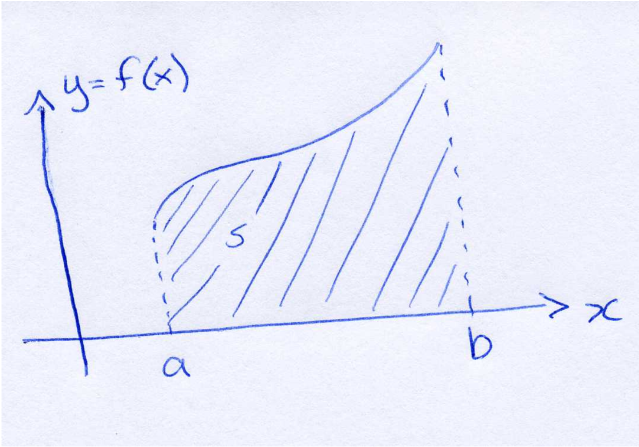
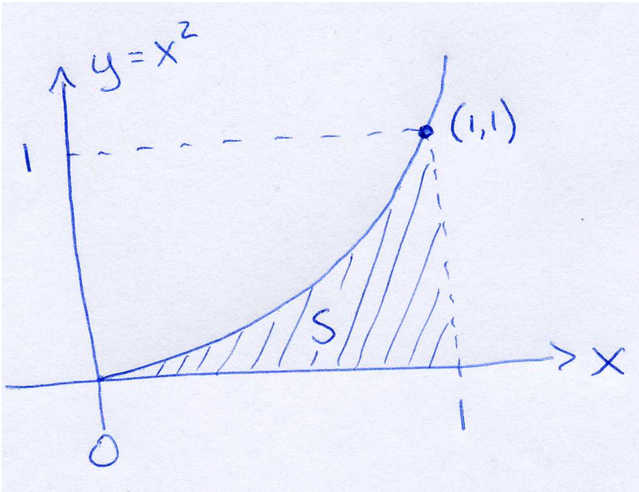
1101 Calculus I 5.1 Areas and Distances

Our goal today: Find the area of the region S that lies under the curve $y = f(x)$ from $x = a$ to $x = b$. For now, we will assume $f(x) \geq 0$.

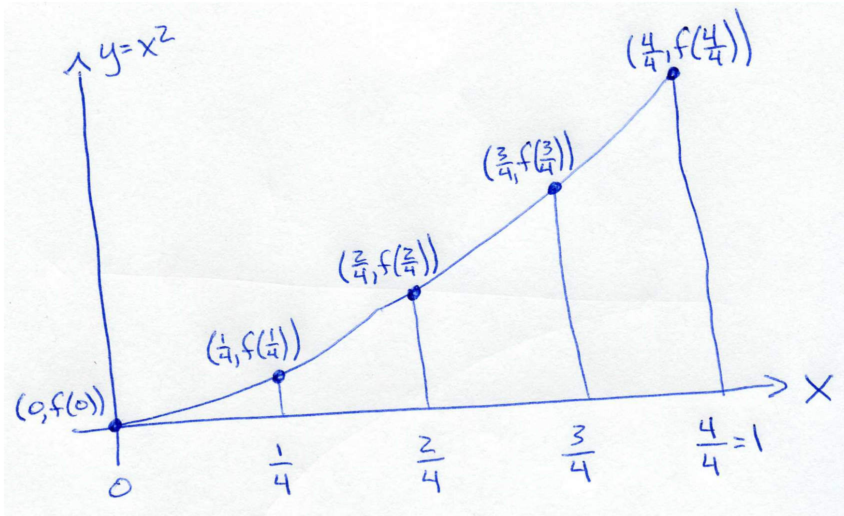


Example $f(x) = x^2$, $a = 0$, $b = 1$.



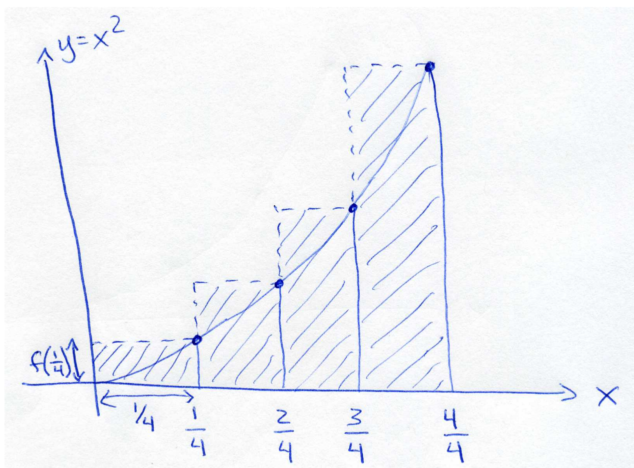
Note Area S is bounded by $0 < S < 1$, since it partially fills a square of area 1.

We can make a better approximation using rectangles! Let's partition the interval $0 \leq x \leq 1$ into 4 subintervals of equal length.



$S = S_1 + S_2 + S_3 + S_4$

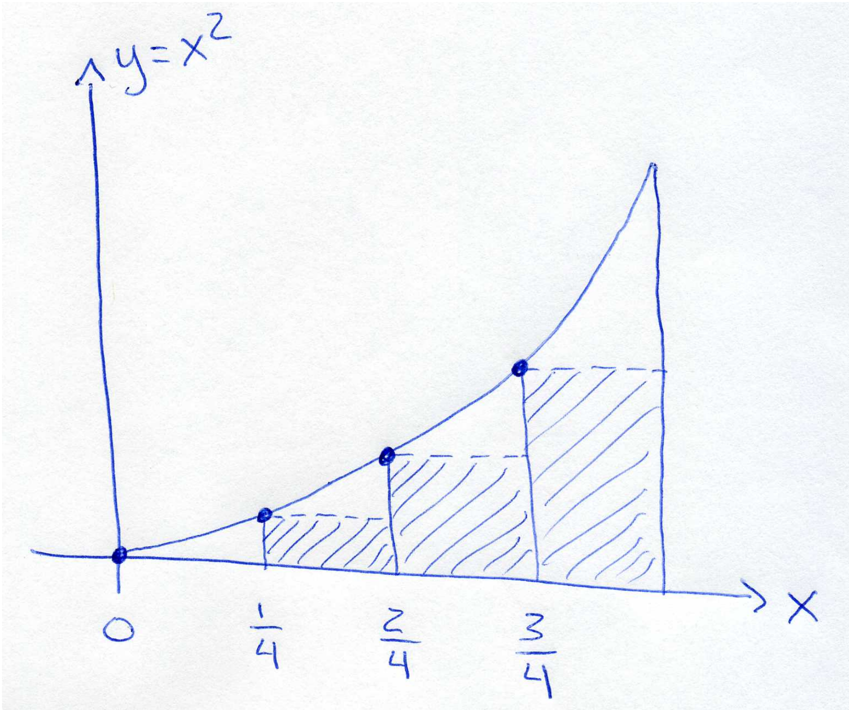
We estimate the area in each of the subintervals as follows:



$$
\begin{aligned}
R_4 &= \frac{1}{4}f\left(\frac{1}{4}\right) + \frac{1}{4}f\left(\frac{2}{4}\right) + \frac{1}{4}f\left(\frac{3}{4}\right) + \frac{1}{4}f\left(\frac{4}{4}\right) \\
&= \frac{1}{4}\left(f\left(\frac{1}{4}\right) + f\left(\frac{2}{4}\right) + f\left(\frac{3}{4}\right) + f\left(\frac{4}{4}\right)\right) \\
&= \frac{1}{4}\left(\left(\frac{1}{4}\right)^2 + \left(\frac{2}{4}\right)^2 + \left(\frac{3}{4}\right)^2 + \left(\frac{4}{4}\right)^2\right) \\
&= 0.46875
\end{aligned}
$$

Note that $S < R_4$. We call it R_4 since it is the right side of the partitions on which the height of the rectangles comes from.

We could also estimate the area in each of the subintervals as follows:



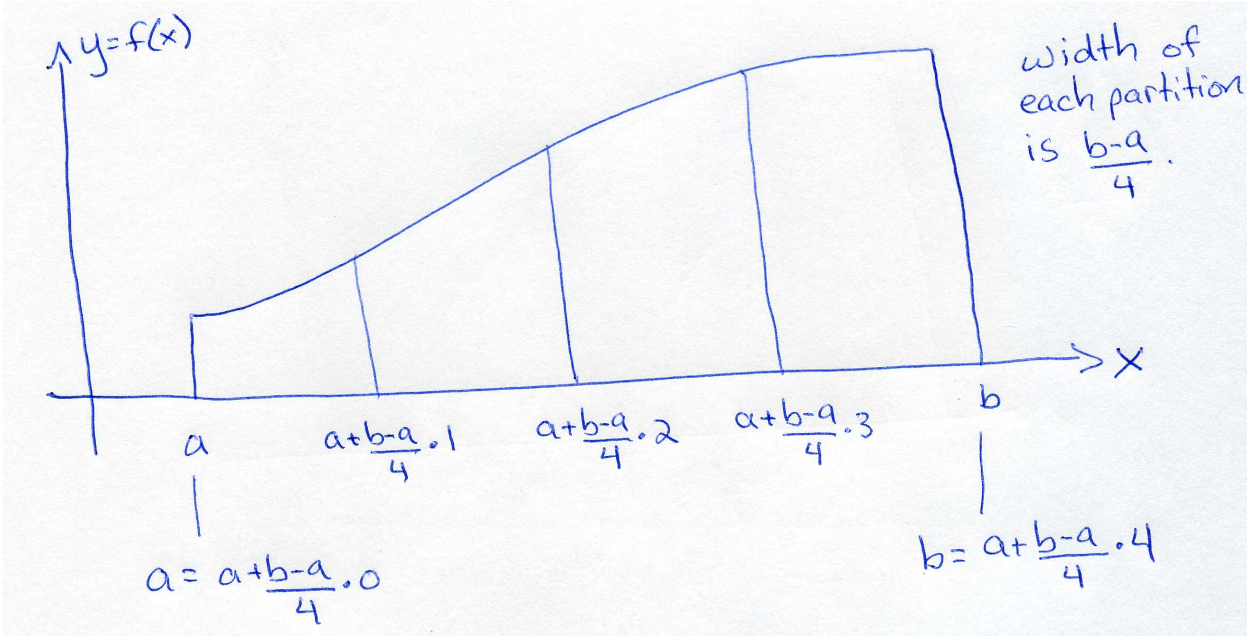
$$
\begin{aligned}
L_4 &= \frac{1}{4}f(0) + \frac{1}{4}f\left(\frac{1}{4}\right) + \frac{1}{4}f\left(\frac{2}{4}\right) + \frac{1}{4}f\left(\frac{3}{4}\right) \\
&= \frac{1}{4}\left(f(0) + f\left(\frac{1}{4}\right) + f\left(\frac{2}{4}\right) + f\left(\frac{3}{4}\right)\right) \\
&= \frac{1}{4}\left((0)^2 + \left(\frac{1}{4}\right)^2 + \left(\frac{2}{4}\right)^2 + \left(\frac{3}{4}\right)^2\right) \\
&= 0.21875
\end{aligned}
$$

Note that $S > L_4$. We call it L_4 since it is the left side of the partitions on which the height of the rectangles comes from.

We know know that $0.21875 < S < 0.46875$.

Note: this bounding property is not always true! A function which is decreasing over the interval will have a different bounding property. A function which is increasing and decreasing over the interval will have no bounding property.

Generalize the result for R_4 to arbitrary function $f(x)$ and arbitrary endpoints $a \le x \le b$ (still use 4 partitions)



Each partition has width $\frac{b-a}{4}$.

$$R_4 = \frac{b-a}{4}\left(f\left(a+\frac{b-a}{4}\cdot 1\right)+f\left(a+\frac{b-a}{4}\cdot 2\right)+f\left(a+\frac{b-a}{4}\cdot 3\right)+f\left(a+\frac{b-a}{4}\cdot 4\right)\right)$$

$$R_4 = \frac{b-a}{4}\sum_{i=1}^{4} f\left(a+\frac{b-a}{4}\cdot i\right)$$

Generalize the result further to n partitions, and get R_n

$$R_n = \frac{b-a}{n}\sum_{i=1}^{n} f\left(a+\frac{b-a}{n}\cdot i\right)$$

Notice the summation is over partitions $1, 2, 3, \ldots, n$. In this notation, we are evaluating f at $a+\frac{b-a}{n}\cdot i$, the right endpoint of the i^{th} partition.

Modify the Above to Use Left EndPoints, and get L_n

To use the left endpoints, we just have to change to evaluating f at the left endpoint of the i^{th} partition, which is $a+\frac{b-a}{n}\cdot(i-1)$

$$L_n = \frac{b-a}{n}\sum_{i=1}^{n} f\left(a+\frac{b-a}{n}\cdot (i-1)\right)$$

To get the actual area, we take the limit as the number of partitions goes to infinity:

$$\begin{aligned} S &= \lim_{n\to\infty} R_n = \lim_{n\to\infty} \frac{b-a}{n} \sum_{i=1}^{n} f\left(a + \frac{b-a}{n} \cdot i\right) \\ S &= \lim_{n\to\infty} L_n = \lim_{n\to\infty} \frac{b-a}{n} \sum_{i=1}^{n} f\left(a + \frac{b-a}{n} \cdot (i-1)\right) \end{aligned}$$

In fact, we can get the area by evaluating the function f at any point x_i^* in each of the i subintervals:

$$S = \lim_{n\to\infty} \frac{b-a}{n} \sum_{i=1}^{n} f\left(x_i^*\right)$$

Another common choice for x_i^* is the midpoint of each partition, which gives M_n:

$$x_i^* = \frac{1}{2}\left[\left(a + \frac{b-a}{n} \cdot i\right) + \left(a + \frac{b-a}{n} \cdot (i-1)\right)\right] = a + \frac{b-a}{n}(i - \frac{1}{2})$$

Summary

The area under the curve $y = f(x)$ where $f(x) \geq 0$ over the interval $a \leq x \leq b$ is S, and is given by

$$\begin{aligned} \text{(in general) } S &= \lim_{n\to\infty} \frac{b-a}{n} \sum_{i=1}^{n} f\left(x_i^*\right) \\ \text{(right hand } x_i^*\text{) } S &= \lim_{n\to\infty} R_n = \lim_{n\to\infty} \frac{b-a}{n} \sum_{i=1}^{n} f\left(a + \frac{b-a}{n} \cdot i\right) \\ \text{(left hand } x_i^*\text{) } S &= \lim_{n\to\infty} L_n = \lim_{n\to\infty} \frac{b-a}{n} \sum_{i=1}^{n} f\left(a + \frac{b-a}{n} \cdot (i-1)\right) \\ \text{(midpoint } x_i^*\text{) } S &= \lim_{n\to\infty} M_n = \lim_{n\to\infty} \frac{b-a}{n} \sum_{i=1}^{n} f\left(a + \frac{b-a}{n} \cdot (i - \frac{1}{2})\right) \end{aligned}$$

To evaluate these quantities, we need to be able to find closed forms for the $\sum_{i=1}^{n} f\left(x_i^*\right)$, which we can only do for certain simple functions f. *Mathematica* knows how to do lots of these.

Note that the $\frac{b-a}{n}$ does not depend on i which is being summed over, which is why I wrote it out in front of the summation (but not in front of the limit, since it does depend on n. You will often see these quantities written as follows:

$$\begin{aligned} \text{(in general) } S &= \lim_{n\to\infty} \sum_{i=1}^{n} f\left(x_i^*\right) \frac{b-a}{n} \\ \text{(right hand } x_i^*\text{) } S &= \lim_{n\to\infty} R_n = \lim_{n\to\infty} \sum_{i=1}^{n} f\left(a + \frac{b-a}{n} \cdot i\right) \frac{b-a}{n} \\ \text{(left hand } x_i^*\text{) } S &= \lim_{n\to\infty} L_n = \lim_{n\to\infty} \sum_{i=1}^{n} f\left(a + \frac{b-a}{n} \cdot (i-1)\right) \frac{b-a}{n} \\ \text{(midpoint } x_i^*\text{) } S &= \lim_{n\to\infty} M_n = \lim_{n\to\infty} \sum_{i=1}^{n} f\left(a + \frac{b-a}{n} \cdot (i - \frac{1}{2})\right) \frac{b-a}{n} \end{aligned}$$

where the $\frac{b-a}{n}$ appears at the end. The reason for this will be apparent when we look at Section 5.2, where we develop a less cumbersome notation to represent the net area under a curve.