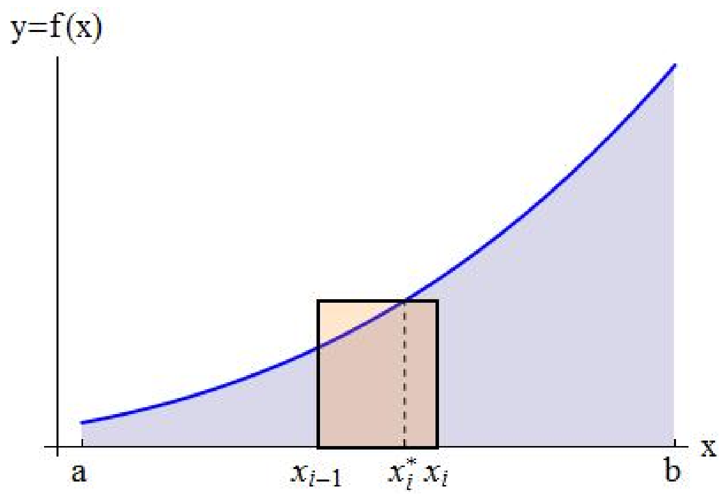
1101 Calculus I 5.2 The Definite Integral

Definition If f is a continuous function defined for $a \leq x \leq b$, we divide the interval $[a, b]$ into n subintervals of equal width $\Delta x = (b - a)/n$.

We let $x_0 = a, x_1, x_2, \ldots, x_n = b$ be the endpoints of these subintervals and we choose sample points $x_1^*, x_2^*, \ldots, x_n^*$ in the subintervals, so $x_i^* \in [x_{i-1}, x_i]$ is in the i^{th} subinterval.

The Definite Integral of f from a to b is $\displaystyle\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x$.

We often draw a single rectangle to represent the creation of the sum $\displaystyle\sum_{i=1}^{n} f(x_i^*)\,\Delta x$.



Notice the height of the rectangle is $f(x_i^*)$, and the width is Δx.

Read $\displaystyle\int_a^b f(x)\,dx$ as "the definite integral of $f(x)$ with respect to x from $x = a$ to $x = b$."

Notice that for both derivative $\dfrac{dy}{dx}$ and definite integral $\displaystyle\int_a^b f(x)\,dx$ we read dx as "with respect to x".

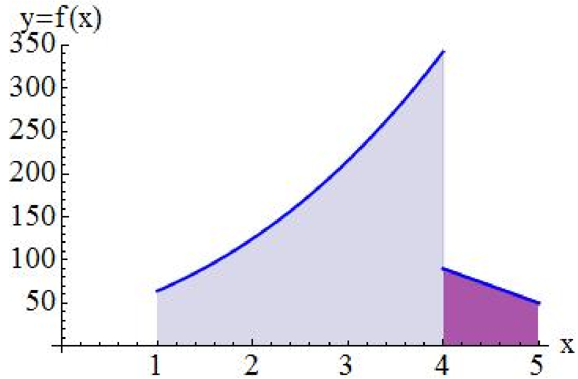
Notation:

- $\int$: the integral sign courtesy of Leibniz.
- $f(x)$ is the integrand.
- a is the lower limit of the integral.
- b is the upper limit of the integral.
- The procedure of calculating an integral is called integration.
- The definite integral is a number, it is not a function of x!! x is a dummy index:

$$\int_a^b f(x)\,dx = \int_a^b f(y)\,dy = \int_a^b f(\heartsuit)\,d\heartsuit$$

If f is continuous, the limit always exists, and does not depend on x_i^*. You can choose any value of x_i^* in the interval!

Question: Does the limit exist for a function x that has a finite jump discontinuity?



The area under the curve exists as a finite number, so the limit exists!

In fact, this example shows us that we could write $\int_1^5 f(x)\,dx = \int_1^4 f(x)\,dx + \int_4^5 f(x)\,dx$ by interpreting the integral as an area. The integral from 1 to 5 is the sum of the area of both shaded regions.

Note this only worked because we had a finite jump discontinuity–vertical asymptotes in the region are considered in Calculus II, so we won't look at anything like that here.

Net Area

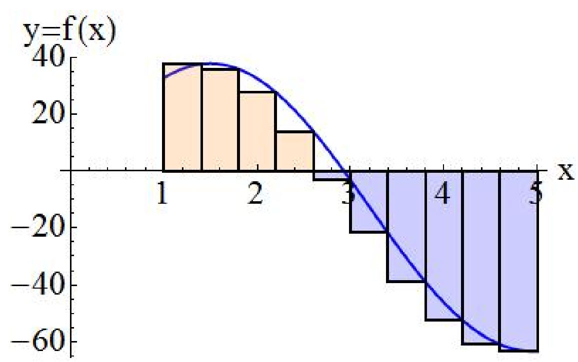
An area above the x-axis is considered a positive area.

An area below the x-axis is considered a negative area.

All area together is considered <u>net area</u>, which means we have the possibility of a negative area. That isn't a problem, since you have seen things like negative money (what you owe on your credit cards counts as a negative towards your net worth).

Mathematically, this makes sense if we consider the sum definition of the integral: $\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x$.

Notice that when the rectangle is below the x-axis the value of $f(x_i^*) < 0$ and the contribution from that subinterval will be negative:

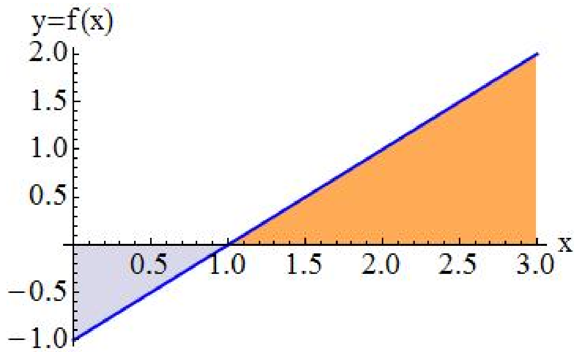


- If you evaluate the integral by sketching the y=integrand and interpreting it as areas, you have to manually insert the subtraction of areas below the x-axis.
- If you evaluate the integral by using the sum $\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x$ or the techniques we will develop in the future, you automatically get the net area.

Therefore, $\int_a^b f(x)\,dx$ represents the net area between $f(x)$ and the x-axis over the interval $x \in [a,b]$.

Example Evaluate the integral by interpreting areas: $\int_0^3 (x-1)\,dx$.

Sketch $y=$integrand, so $y = x-1$ and remember that areas below the x-axis are negative.



$$\int_0^3 (x-1)\,dx = (\text{area of large orange triangle}) - (\text{area of small blue triangle}) = \frac{1}{2}(3-1)(2) - \frac{1}{2}(1)(1) = \frac{3}{2}$$

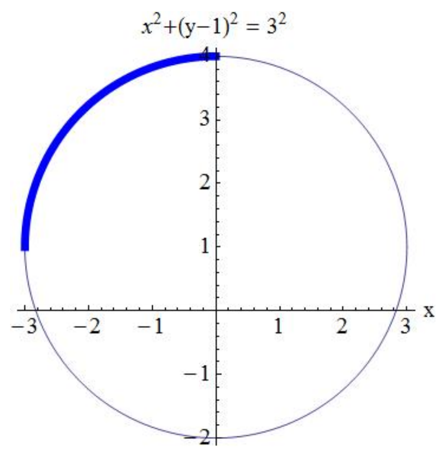
Example Evaluate the integral by interpreting areas: $\int_{-3}^0 (1+\sqrt{9-x^2})dx$.

We need to sketch the integrand, which is $y = 1 + sqrt 9 - x^2$. Do not use the techniques of sketching we just learned in Chapter 4! To interpret as an area we have to be able to geometrically determine the area, so this can't be much more complicated than a circle.

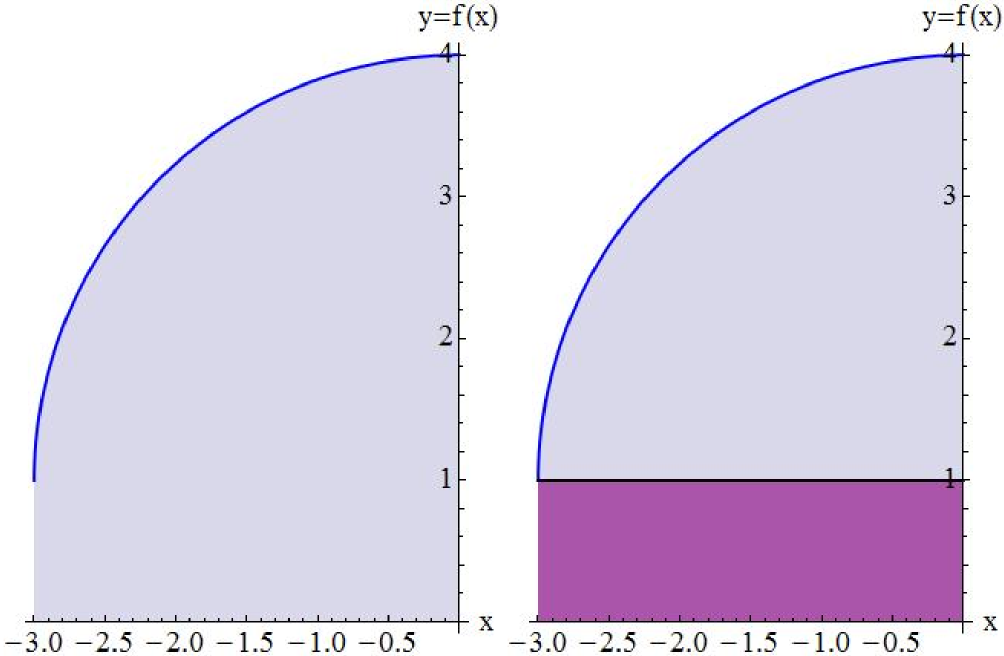
Note: Since we have + in front of the square root, we have only the top half of the circle in our function.

$$\begin{aligned} y &= 1+\sqrt{9-x^2} \\ y-1 &= \sqrt{9-x^2} \\ (y-1)^2 &= 9-x^2 \\ x^2+(y-1)^2 &= 3^2 \end{aligned}$$

So this is a circle of radius 3 centered at $(0,1)$. The top half between $x=-3$ and $x=0$ are all we care about for this particular integral (thick line).



The limits of integration are from $x = -3$ to $x = 0$, so the area the integral represents is the area between $f(x)$ and the x-axis for $x \in [-3, 0]$:



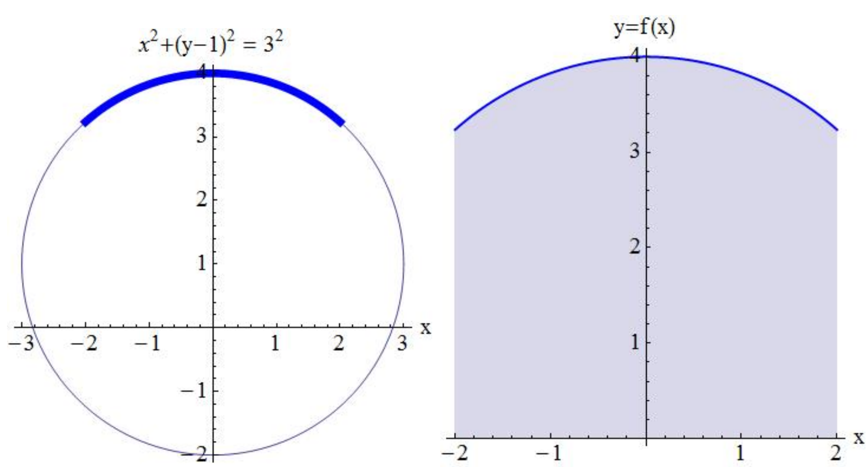
First plot: Shaded area is the value of $\int_{-3}^{0} (1 + \sqrt{9 - x^2})dx$.

Second plot: shaded area from first plot split into a rectangle and one quarter of the area of circle of radius 3. This is what we use to evaluate the integral. We add the two areas since both are above the x-axis.

$$\int_{-3}^{0} (1 + \sqrt{9 - x^2})dx = \frac{1}{4}(\text{area of circle of radius 3}) + (\text{area of rectangle}) = \frac{1}{4}\pi(3)^2 + (1)(3) = \frac{9}{4}\pi + 3$$

Note that to evaluate this integral in ways other than interpreting as areas requires trigonometric substitution, which you would learn in Calculus II.

FYI, if we instead had $\int_{-2}^{2} (1 + \sqrt{9 - x^2})dx$, we could interpret as areas as follows:



We could not get the areas through geometry since the areas are not a simple fraction of the area of a circle. We would have to use the techniques from Calculus II to evaluate it, since interpreting as areas is unworkable.

Properties of Integrals You can prove all of these from the Riemann sum definition of the integral.

1. $\int_b^a f(x)dx = -\int_a^b f(x)dx$
2. $\int_a^a f(x)dx = 0$
3. $\int_a^b cdx = c(b-a)$ (c is a constant)
4. $\int_a^b [f(x)+g(x)]dx = \int_a^b f(x)dx + \int_a^b g(x)dx$
5. $\int_a^b cf(x)dx = c\int_a^b f(x)dx$ (c is a constant)
6. $\int_a^c f(x)dx + \int_c^b f(x)dx = \int_a^b f(x)dx$

Comparison Properties of Integrals

1. $f(x) \geq 0$ for $a \leq x \leq b \longrightarrow \int_b^a f(x)dx \geq 0$
2. $f(x) \geq g(x)$ for $a \leq x \leq b \longrightarrow \int_b^a f(x)dx \geq \int_a^b f(x)dx$
3. $m \leq f(x) \leq M$ for $a \leq x \leq b \longrightarrow m(b-a) \leq \int_b^a f(x)dx \leq M(b-a)$

Example Given $\int_4^9 \sqrt{x}\,dx = 38/3$, what is $\int_9^4 \sqrt{t}\,dt$?

$$\int_9^4 \sqrt{t}\,dt = -\int_4^9 \sqrt{t}\,dt = -\frac{38}{3}$$

Example Write $\int_2^{10} f(x)\,dx - \int_2^7 f(x)\,dx$ as $\int_a^b f(x)\,dx$.

$$\begin{aligned}
\int_2^{10} f(x)\,dx - \int_2^7 f(x)\,dx &= -\int_{10}^2 f(x)\,dx - \int_2^7 f(x)\,dx \\
&= -\int_{10}^7 f(x)\,dx \\
&= \int_7^{10} f(x)\,dx
\end{aligned}$$

Example Express the limit as a definite integral:

$$\lim_{n\to\infty}\sum_{i=1}^n \frac{i^4}{n^5} = \lim_{n\to\infty}\frac{b-a}{n}\sum_{i=1}^n f(a+\frac{b-a}{n}i) = \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^n \left(\frac{i}{n}\right)^4$$

Comparing to determine a, b, f, we see: $a = 0$, $b = 1$, $f(x) = x^4$ so we have $\int_0^1 x^4 dx$.