1101 Calculus I 5.3 The Fundamental Theorem of Calculus

So far, we have looked at

1. Differential Calculus (interpretation: tangents)
2. Integral Calculus (interpretation: net areas)

Note that so far, we have seen that

- $\frac{d}{dx}[f(x)]$ can always be evaluated (provided the derivative exists)
- $\int_a^b f(x)\,dx$ can be approximated using Riemann Sums for specific n: $\sum_{i=1}^{n} f(x_i^*)\Delta x$.
- $\int_a^b f(x)\,dx$ can be determined exactly in simple cases if we can sketch the integrand and determine the net area.
- We need techniques to evaluate an integral exactly in other situations, and the FTC is the first of these techniques.

The Fundamental Theorem of Calculus relates the apparently different concepts of derivative and integral. It will allow us to compute integrals without using Riemann sums, or interpreting as an area.

Motivation

The full proof is in the text, but requires some concepts that we have skipped. This is a motivation of the FTC Part 1.

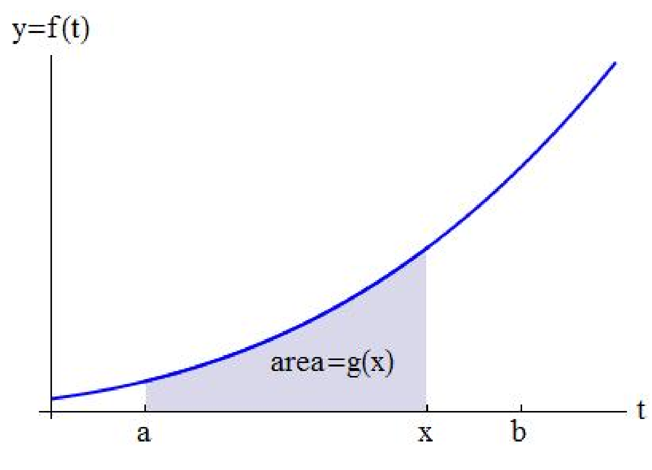
Consider the following function:

$$g(x) = \int_a^x f(t)\,dt, \quad a \le x \le b$$

Note: g is only a function of x, the upper limit in the integral. The t is a dummy index.

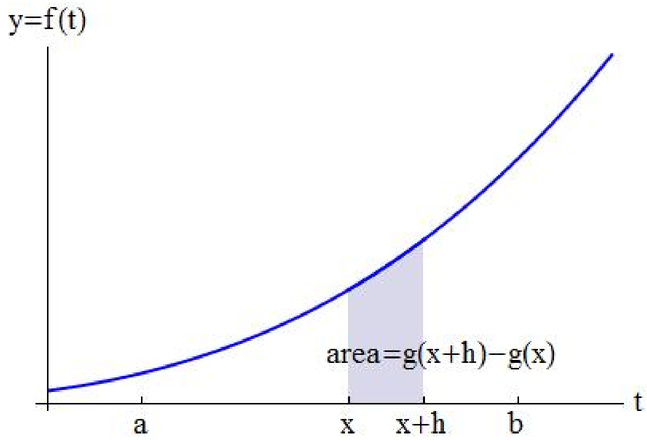
Question: What is the relation between g and f?

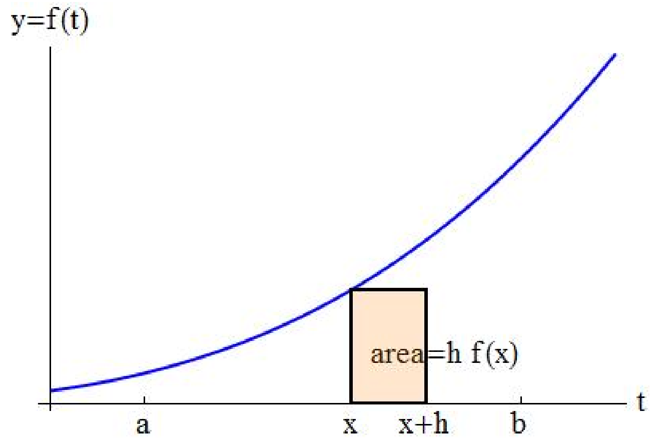
If $f(x) \ge 0$, then we can interpret $g(x) = \int_a^x f(t)\,dt$ as the area under the graph of f.



Now, let's compute $g'(x)$ from the definition of derivative.

If $h > 0$, then $g(x+h) - g(x)$ is obtained by subtracting areas:





The two areas are approximately equal if h is small.

$$hf(x) \sim g(x+h) - g(x)$$

$$f(x) \sim \frac{g(x+h) - g(x)}{h}$$

Take $h \to 0$, and the approximation becomes an equality:

$$f(x) = \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} = g'(x)$$

The FTC Part 1 using Liebniz notation: If $g = \int_a^x f(t)\,dt$ then $g'(x) = \frac{d}{dx}\int_a^x f(t)\,dt = f(x)$

Note:

- the lower limit must be a constant. If it isn't, use properties of derivatives to make it a constant.
- the upper limit must just be an x. If it is a more complicated function of x you have to use a chain rule of derivatives.

Example Find the derivative of $g(x) = \int_4^x \sin t\,dt$.

$$g'(x) = \frac{d}{dx}\int_4^x \sin t\,dt = \sin x$$

Example Find $\frac{d}{dx}\int_1^{x^4} \sec t\,dt$.

The function is actually a function of x^4, so this requires the use of the chain rule. Here is how we can find the derivative:

$$\begin{aligned}\frac{d}{dx}\int_1^{x^4} \sec t\,dt &= \frac{d}{dx}\int_1^{u} \sec t\,dt, \quad u = x^4\\ &= \frac{d}{du}\int_1^{u} \sec t\,dt \cdot \frac{du}{dx}, \quad \text{chain rule of derivatives}\\ &= \sec u \cdot (4x^3)\\ &= 4x^3 \sec(x^4)\end{aligned}$$

Example Find $\dfrac{d}{dx}\displaystyle\int_x^{x^2} t\,dt$.

The integral does not have a lower limit that is a constant. We can fix that as follows:

$$\begin{aligned}
\frac{d}{dx}\int_x^{x^2} t\,dt &= \frac{d}{dx}\int_x^{a} t\,dt + \frac{d}{dx}\int_a^{x^2} t\,dt \quad a \text{ some constant, using properties of integrals} \\
&= -\frac{d}{dx}\int_a^{x} t\,dt + \frac{d}{dx}\int_a^{x^2} t\,dt \\
&= -x + \frac{d}{du}\int_a^{u} t\,dt \cdot \frac{du}{dx} \quad u = x^2 \quad \text{chain rule of derivatives for second term} \\
&= -x + u \cdot (2x) \\
&= -x + 2x^3
\end{aligned}$$

What if we integrate first, then differentiate? That's the FTC Part 2.

The Fundamental Theorem of Calculus Part 2 If f is continuous on $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

where F is any antiderivative of f.

WOW!! The integral, which was defined as that Riemann sum over a partitioned interval, can be found by evaluating the antiderivative at the endpoints only!

Motivation

$$\begin{aligned}
\int_a^b f(x)\,dx &= \int_a^b \frac{dF}{dx}\,dx \quad \text{since } F \text{ is an antiderivative of } f\text{, we have } f = \frac{dF}{dx} \\
&= \int_a^b dF \quad \text{cancel differential } dx
\end{aligned}$$

Now let's think about this a bit. Integration is adding up stuff. The differential dF is a small bit of F. So conceptually the integral above is adding up small bits of F to get F!

$$\begin{aligned}
\int_a^b f(x)\,dx &= \int_a^b dF \\
&= F\Big|_a^b = F(b) - F(a)
\end{aligned}$$

The last line is the notation we use to show how the result of the integral is evaluated at upper limit minus the result at the lower limit.

To use the FTC Part 2, you have to be able to determine an antiderivative of the integrand.

Example Evaluate $\displaystyle\int_1^3 e^x\,dx$.

An antiderivative of e^x is e^x. We can work with this.

$$\int_1^3 e^x\,dx = e^x\Big|_1^3 = e^3 - e^1.$$

Example Find the area under the parabola $y = x^2$ from $x = 0$ to $x = 1$.

Since $f(x) = x^2 \geq 0$ for $x \in [0, 1]$, we can interpret the area as the integral:

$$\text{Area } = \int_0^1 x^2\, dx = \left.\frac{x^3}{3}\right|_0^1 = \frac{1^3}{3} - \frac{0^3}{3} = \frac{1}{3}.$$

Example

$$\int_3^6 \frac{dx}{x} = \int_3^6 \frac{1}{x} dx$$

An antiderivative of the integrand $f(x) = 1/x$ is $F(x) = \ln|x|$.
However, $x > 0$, so we can use $\ln x$.

$$\int_3^6 \frac{dx}{x} = \ln x\Big|_3^6 = \ln 6 - \ln 3 = \ln\left(\frac{6}{3}\right) = \ln 2$$

Example Evaluate $\dfrac{d}{dx}\displaystyle\int_x^{x^2} t\, dt$ by evaluating the integral and then differentiating.

An antiderivative of t is $\dfrac{t^2}{2}$.

$$\begin{aligned} \int_x^{x^2} t\, dt &= \left.\frac{t^2}{2}\right|_x^{x^2} \\ &= \frac{(x^2)^2}{2} - \frac{x^2}{2} = \frac{x^4}{2} - \frac{x^2}{2} \\ \frac{d}{dx}\int_x^{x^2} t\, dt &= \frac{d}{dx}\left[\frac{x^4}{2} - \frac{x^2}{2}\right] = 2x^3 - x \end{aligned}$$

which agrees with what we found when we did this earlier using FTC Part 1. Note that the FTC Part 1 is extremely useful at times because you can use it even when you can't find an antiderivative of the integrand:

$$\frac{d}{dx}\int_a^x \cos(e^{t^{23}-4})\, dt = \cos(e^{x^{23}-4})$$

Example Does $\displaystyle\int_{-1}^3 \frac{1}{x^2} dx$ make sense? NO! It does not make sense since the integrand is not continuous over the interval $[-1, 3]$. This is an improper integral that you would study in Calculus II.

Summary

The Fundamental Theory of Calculus (both parts): Suppose f is continuous on $[a, b]$. Then

1. If $g(x) = \displaystyle\int_a^x f(t)\, dt$, then $g'(x) = f(x)$.
2. $\displaystyle\int_a^b f(x)\, dx = F(b) - F(a)$ where F is any antiderivative of f, that is $F'(x) = f(x)$.

The Derivative and Integral as Inverses

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

If f is integrated and then differentiated, we get f back.

$$\int_a^b F'(x)\,dx = \int_a^b \frac{d}{dx}F(x)dx = F(b) - F(a)$$

If F is differentiated and then integrated, we get F back in the form $F(b) - F(a)$.

Example Integrate $f(x)$ for $x \in [-\pi, \pi]$ where

$$f(x) = \begin{cases} x+2 & x \le 0 \\ \sin x & x > 0 \end{cases}$$

$$\begin{aligned}
\int_{-\pi}^{\pi} f(x)\,dx &= \int_{-\pi}^{0} f(x)\,dx + \int_0^{\pi} f(x)\,dx \\
&= \int_{-\pi}^{0} (x+2)\,dx + \int_0^{\pi} \sin x\,dx \\
&= \left(\frac{x^2}{2} + 2x\right)_{-\pi}^{0} + \Big(-\cos x\Big)_0^{\pi} \\
&= \left(\frac{0^2}{2} + 2(0)\right) - \left(\frac{(-\pi)^2}{2} + 2(-\pi)\right) - \cos\pi + \cos 0 \\
&= -\frac{\pi^2}{2} + 2\pi + 2
\end{aligned}$$