Example A particle moves along a line so that $v(t) = 3t - 5$ m/s. Positive displacement is measured to the right.
a) Find the displacement of the particle during the time period $0 \le t \le 3$.
b) Find the distance traveled during this time period.

$$\begin{aligned}
\text{displacement} &= s(3) - s(0) \\
&= \int_0^3 v(t)\,dt \\
&= \int_0^3 (3t-5)\,dt \\
&= \left(\frac{3t^2}{2} - 5t\right)_0^3 \\
&= \left(\frac{3(3)^2}{2} - 5(3)\right) - \left(\frac{3(0)^2}{2} - 5(0)\right) \\
&= -\frac{3}{2}\text{ m}
\end{aligned}$$

The particle moved 3/2 m to the left (because of the minus sign).

$$\begin{aligned}
\text{distance traveled} &= \int_0^3 |v(t)|\,dt \\
&= \int_0^3 |3t-5|\,dt
\end{aligned}$$

We need to work out the absolute value:

$$\begin{aligned}
|3t-5| &= \begin{cases} (3t-5) & \text{if } 3t-5 \ge 0 \\ -(3t-5) & \text{if } 3t-5 < 0 \end{cases} \\
&= \begin{cases} (3t-5) & \text{if } t \ge \frac{5}{3} \\ -(3t-5) & \text{if } t < \frac{5}{3} \end{cases} \\
\text{distance traveled} &= \int_0^3 |3t-5|\,dt \\
&= \int_0^{5/3} |3t-5|\,dt + \int_{5/3}^3 |3t-5|\,dt \\
&= \int_0^{5/3} (-(3t-5))\,dt + \int_{5/3}^3 (3t-5)\,dt \\
&= -\left(\frac{3t^2}{2} - 5t\right)_0^{5/3} + \left(\frac{3t^2}{2} - 5t\right)_{5/3}^3 \\
&= -\left(\frac{3(5/3)^2}{2} - (5)\frac{5}{3}\right) + \left(\frac{3(0)^2}{2} - 5(0)\right) + \left(\frac{3(3)^2}{2} - 5(3)\right) - \left(\frac{3(5/3)^2}{2} - (5)\frac{5}{3}\right) \\
&= \frac{41}{6}\text{ m}
\end{aligned}$$

The total distance traveled by the particle is 41/6 m.

Example Find the distance traveled by a particle during the time from $t = 0$ to $t = 10$ if the particle moves with the acceleration $a(t) = t + 4$ and the initial velocity is $v(0) = 5$.

The velocity will be given by the integral of the acceleration:

$$v(t) = \int a(t)\, dt + c$$

$$v(t) = \frac{t^2}{2} + 4t + c$$

Use the condition to determine the constant c:

$$v(0) = 0 + 0 + c = 5$$

$$v(t) = \frac{t^2}{2} + 4t + 5 \text{ m/s}$$

The distance traveled is given by:

$$\int_0^{10} |v(t)|\, dt = \int_0^{10} \left| \frac{t^2}{2} + 4t + 5 \right| dt$$

The integrand is positive in the region $0 \leq t \leq 10$, so the absolute value can be replaced with the function:

$$\left| \frac{t^2}{2} + 4t + 5 \right| = \frac{t^2}{2} + 4t + 5$$

Aside: You can show this by working out the roots of the quadratic and seeing that they are both less than zero.

$$\text{roots} \;=\; \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-4 \pm \sqrt{16 - 10}}{1} = -4 \pm \sqrt{6} < 0$$

$$\begin{aligned}
\text{Distance Traveled} \quad &= \int_0^{10} |v(t)|\, dt \\
&= \int_0^{10} (\frac{t^2}{2} + 4t + 5)\, dt \\
&= \left(\frac{t^3}{6} + 2t^2 + 5t \right)_0^{10} \\
&= \left(\frac{10^3}{6} + 2(10)^2 + 5(10) \right) - \left(\frac{0^3}{6} + 2(0)^2 + 5(0) \right) \\
&= \frac{1000}{6} + 200 + 50 = \frac{1250}{3}
\end{aligned}$$

The distance traveled by the particle is 1250/3 m.