This is a set of practice test problems for Chapter 3. This is in **no way** an inclusive set of problems–there can be other types of problems on the actual test. The solutions are what I would accept on a test, but you may want to add more detail, and explain your steps with words–remember, I am interested in the process you use to solve problems!

There will be five problems on the test. Most will involve more than one part. You will have 100 minutes to complete the test. You may not use *Mathematica* or calculators on this test.

1. Find the equation of the tangent line to the curve $y = x + \cos x$ at the point $(0, 1)$.

2. Find the x–coordinates on the curve $x^3 + y^3 = 6xy$ where the tangent line is horizontal.

3. Prove the Power Rule using logarithmic differentiation.

The Power Rule: If n is any real number and $f(x) = x^n$, then $f'(x) = nx^{n-1}$.

4. Prove the Sum Rule,

$$\frac{d}{dx}[f(x) + g(x)] = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)].$$

5. Find dy/dx if

$$\frac{1}{x} + \frac{1}{y} = 1.$$

6. Find y' if $y = \arctan x^x$.

7. Find the second derivative y'' using implicit differentiation if $x^3 + y^3 = 10$. Note $y'' = \frac{d}{dx}[\frac{dy}{dx}]$.

8. Find y' if $y = (\sec x)^x + e^{\sin x}$.

9. Use logarithmic differentiation to find the derivative of the function $y = (2x + 1)^5(x^4 - 3)^6$.

10. Given $y = (\sin x)^x$, show $y' = (\sin x)^x(\ln(\sin x) + x \cot x)$.

11. A man starts walking north at 4 ft/s from a point P. Five minutes later a woman starts walking south at 5 ft/s from a point 500ft due east of P. At what rate are the people moving apart 15 mins after the woman started walking?

12. A ladder 12 ft long rests against a vertical wall. If the bottom of the ladder slides away from the wall at a rate of 2 ft/s, how fast is the top of the ladder sliding down the wall when the bottom of the ladder is 6 ft from the wall?

13. Show that

$$\frac{d}{dx}[\arctan x] = \frac{1}{1 + x^2}.$$

Solutions

Problem 1. Statements:
The slope of the tangent line is the derivative of the function.
We want the equation of the tangent line, so our answer will look like $y - y_0 = m(x - x_0)$.
The point we are interested in is $(x_0, y_0) = (0, 1)$, which has $x = 0$.
We want to find the derivative $f'(0) = m$.
We need to define $f(x) = x + \cos x$.
Our answer will look like $y - 1 = f'(0)(x - 0)$.

$$\begin{aligned} f'(x) &= \frac{d}{dx}[x + \cos x] \\ &= 1 - \sin[x] \\ f'(0) &= 1 - \sin 0 = 1 \end{aligned}$$

The equation of the tangent line to the curve at (0,1) is

$$\begin{aligned} y - 1 &= f'(0)(x - 0) \\ y - 1 &= (1)x \\ y &= x + 1 \end{aligned}$$

Problem 2. Statements:
The tangent line is horizontal means $f'(x) = 0$.
However, $x^3 + y^3 = 6xy$ is an implicit function.
We will need to find the x-coordinates where $y' = dy/dx = 0$ (we cannot use $f'(x)$, since the function is implicit).

$$\begin{aligned} x^3 + y^3 &= 6xy \\ \frac{d}{dx}\left[x^3 + y^3\right. &= \left.6xy\right] \quad \text{implicitly differentiate} \\ \frac{d}{dx}\left[x^3\right] + \frac{d}{dx}\left[y^3\right] &= \frac{d}{dx}[6xy] \\ 3x^2 + \frac{d}{dy}\left[y^3\right] \cdot \frac{dy}{dx} &= 6y\frac{d}{dx}[x] + 6x\frac{d}{dx}[y] \\ 3x^2 + 3y^2\frac{dy}{dx} &= 6(1)y + 6x\frac{dy}{dx} \\ (3y^2 - 6x)\frac{dy}{dx} &= 6y - 3x^2 \\ \frac{dy}{dx} &= \frac{6y - 3x^2}{3y^2 - 6x} \end{aligned}$$

Now, if $dy/dx = 0$, we must have $2y - x^2 = 0$, or $y = x^2/2$.
To get the x-coordinates for the points on the curve, substitute into the equation:

$$\begin{aligned} x^3 + y^3 &= 6xy \\ x^3 + \left(\frac{x^2}{2}\right)^3 &= 6x\left(\frac{x^2}{2}\right) \\ x^3 + \frac{x^6}{8} &= 3x^3 \\ 8x^3 + x^6 &= 24x^3 \\ x^6 &= 16x^3 \\ x^3 &= 16 \end{aligned}$$

The only real valued solutions are $x = 0$ and $x = 16^{1/3}$.

Problem 3. Use logarithmic differentiation:

$$\begin{aligned}
y &= x^n \\
\ln[y &= x^n] \\
\ln[y] &= \ln[x^n] \\
\ln[y] &= n\ln[x] \\
\frac{d}{dx}[\ln[y] &= n\ln[x]] \quad \text{implicitly differentiate} \\
\frac{d}{dx}\ln[y] &= n\frac{d}{dx}\ln[x] \\
\frac{d}{dy}\ln[y]\cdot\frac{dy}{dx} &= n\frac{1}{x} \quad \text{chain rule} \\
\frac{1}{y}\frac{dy}{dx} &= n\frac{1}{x} \\
\frac{dy}{dx} &= n\frac{y}{x} \\
&= n\frac{x^n}{x} \\
&= nx^{n-1} \\
\frac{d}{dx}\left[x^n\right] &= nx^{n-1}
\end{aligned}$$

Problem 4. Let $F(x) = f(x) + g(x)$.

$$\begin{aligned}
F'(x) &= \lim_{h\to 0}\frac{F(x+h)-F(x)}{h} \\
&= \lim_{h\to 0}\frac{f(x+h)+g(x+h)-f(x)-g(x)}{h} \\
&= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} + \lim_{h\to 0}\frac{g(x+h)-g(x)}{h} \\
F'(x) &= f'(x) + g'(x) \\
\frac{d}{dx}[f(x)+g(x)] &= \frac{d}{dx}f(x) + \frac{d}{dx}g(x) \quad \text{(Leibniz notation)}
\end{aligned}$$

Problem 5. Use implicit differentiation (you can do this problem other ways):

$$\begin{aligned}
\frac{1}{x}+\frac{1}{y} &= 1 \\
\frac{d}{dx}[\frac{1}{x}+\frac{1}{y} &= 1] \quad \text{implicitly differentiate} \\
\frac{d}{dx}[x^{-1}] + \frac{d}{dx}[y^{-1}] &= \frac{d}{dx}[1] \\
-x^{-2} + \frac{d}{dy}[y^{-1}]\cdot\frac{dy}{dx} &= 0 \quad \text{chain rule (and others)} \\
-x^{-2} - y^{-2}\frac{dy}{dx} &= 0 \\
\frac{dy}{dx} &= -\frac{y^2}{x^2}
\end{aligned}$$

We have an implicit expression for the derivative, which is good enough since the original equation was given implicitly.

Problem 6. If $y = \arctan(x^x)$, find y'.

$$\begin{aligned} y' = \frac{dy}{dx} &= \frac{d}{dx}[y] = \frac{d}{dx}[\arctan(x^x)] \\ &= \frac{d}{dx}[\arctan(u)], \quad u = x^x \\ &= \frac{d}{du}[\arctan(u)] \cdot \frac{du}{dx} \quad \text{chain rule} \end{aligned}$$

So no we need two derivatives. If you need to, you can work out the derivative of $\arctan u$ with respect to u using logarithmic differentiation, or if you have it memorized just write it down.

$$\frac{d}{du}[\arctan u] = \frac{1}{1+u^2}$$

The other derivative requires logarithmic differentiation, so let $w = x^x$:

$$\begin{aligned} &\ln[w = x^x] \\ &\ln w = x \ln x \\ &\frac{d}{dx}[\ln w = x \ln x] \\ &\frac{d}{dw}[\ln w]\frac{dw}{dx} = \frac{d}{dx}[x \ln x] \\ &\frac{1}{w}\frac{dw}{dx} = x\frac{d}{dx}[\ln x] + \ln x \frac{d}{dx}[x] \\ &\frac{dw}{dx} = w\left(x\frac{1}{x} + \ln x\right) \\ &\frac{dw}{dx} = x^x(1 + \ln x) \end{aligned}$$

Put it all back together, and substitute back $u = x^x$:

$$y' = \frac{1}{1+u^2} \cdot x^x(1+\ln x) = \frac{x^x(1+\ln x)}{1+x^{2x}}$$

Problem 7. Implicitly differentiate:

$$\begin{aligned} x^3 + y^3 &= 10 \\ \frac{d}{dx}[x^3 + y^3 &= 10] \\ \frac{d}{dx}[x^3] + \frac{d}{dx}[y^3] &= \frac{d}{dx}[10] \\ 3x^2 + \frac{d}{dy}[y^3] \cdot \frac{dy}{dx} &= 0 \\ 3x^2 + 3y^2\frac{dy}{dx} &= 0 \\ \frac{dy}{dx} &= -\frac{x^2}{y^2} \\ \frac{d^2y}{dx^2} &= -\frac{d}{dx}\left[\frac{x^2}{y^2}\right] \\ &= -\left(\frac{y^2\frac{d}{dx}[x^2] - x^2\frac{d}{dx}[y^2]}{y^4}\right) \end{aligned}$$

$$
\begin{aligned}
&= -\left(\frac{y^2(2x) - x^2\dfrac{d}{dy}[y^2]\cdot\dfrac{dy}{dx}}{y^4}\right)\\
&= -\left(\frac{2xy^2 - x^2(2y)\cdot\left(\dfrac{-x^2}{y^2}\right)}{y^4}\right)\\
&= -\left(\frac{2xy^2 + \left(\dfrac{2x^4}{y}\right)}{y^4}\right) = -\frac{2x}{y^2} - \frac{2x^4}{y^5}
\end{aligned}
$$

Problem 8. Find y' if $y = (\sec x)^x + e^{\sin x}$.

$$
\begin{aligned}
\frac{dy}{dx} &= \frac{d}{dx}[y] = \frac{d}{dx}[(\sec x)^x + e^{\sin x}]\\
&= \frac{d}{dx}[(\sec x)^x] + \frac{d}{dx}[e^{\sin x}]
\end{aligned}
$$

The first derivative will require logarithmic differentiation; the second derivative requires parts.

$$
\begin{aligned}
w &= (\sec x)^x\\
\ln w &= \ln(\sec x)^x\\
\ln w &= x\ln\sec x\\
\frac{d}{dx}[\ln w] &= \frac{d}{dx}[x\ln\sec x]\\
\frac{d}{dw}[\ln w]\cdot\frac{dw}{dx} &= x\frac{d}{dx}[\ln\sec x] + \ln\sec x\frac{d}{dx}[x]\\
\frac{1}{w}\cdot\frac{dw}{dx} &= x\frac{d}{dx}[\ln u] + \ln\sec x(1) \quad u = \sec x\\
\frac{1}{w}\cdot\frac{dw}{dx} &= x\frac{d}{du}[\ln u]\cdot\frac{du}{dx} + \ln\sec x\\
\frac{1}{w}\cdot\frac{dw}{dx} &= x\cdot\frac{1}{u}\cdot(\sec x\tan x) + \ln\sec x\\
\frac{dw}{dx} &= (\sec x)^x\left(\frac{x\sec x\tan x}{\sec x} + \ln\sec x\right)
\end{aligned}
$$

That was the hard one!

$$
\begin{aligned}
\frac{d}{dx}[e^{\sin x}] &= \frac{d}{dx}[e^u] \quad u = \sin x\\
&= \frac{d}{du}[e^u]\cdot\frac{du}{dx}\\
&= e^u(\cos x)\\
&= e^{\sin x}\cos x
\end{aligned}
$$

Put it all back together:

$$
\frac{dy}{dx} = \frac{d}{dx}[(\sec x)^x] + \frac{d}{dx}[e^{\sin x}] = (\sec x)^x\left(\frac{x\sec x\tan x}{\sec x} + \ln\sec x\right) + e^{\sin x}\cos x
$$

Problem 9. Take the natural logarithm:

$$
y = (2x+1)^5(x^4-3)^6
$$

$$\begin{aligned}
\ln y &= \ln[(2x+1)^5(x^4-3)^6] \\
&= \ln[(2x+1)^5] + \ln[(x^4-3)^6] \\
\ln y &= 5\ln(2x+1) + 6\ln(x^4-3)
\end{aligned}$$

Implicitly differentiate:

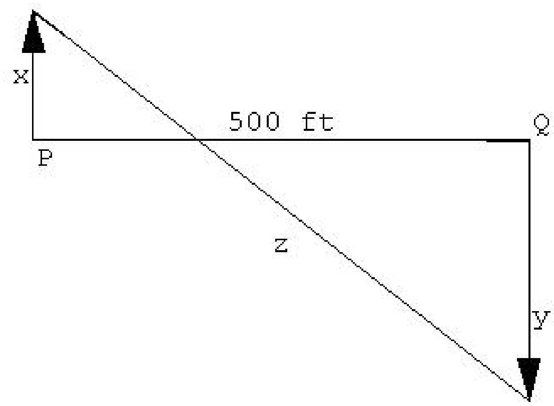
$$\begin{aligned}
\frac{d}{dx}\ln y &= \frac{d}{dx}[5\ln(2x+1) + 6\ln(x^4-3)] \\
\frac{d}{dy}\ln y \cdot \frac{dy}{dx} &= 5\frac{d}{dx}[\ln(2x+1)] + 6\frac{d}{dx}[\ln(x^4-3)] \\
\frac{1}{y}\frac{dy}{dx} &= 5\frac{\frac{d}{dx}[2x+1]}{2x+1} + 6\frac{\frac{d}{dx}[x^4-3]}{x^4-3} \\
\frac{dy}{dx} &= y\left(5\frac{2}{2x+1} + 6\frac{4x^3}{x^4-3}\right) \\
\frac{dy}{dx} &= (2x+1)^5(x^4-3)^6\left(\frac{10}{2x+1} + \frac{24x^3}{x^4-3}\right)
\end{aligned}$$

We backsubstitute for y since the original equation was explicit, so our answer should be explicit as well.

Problem 10. Since we have a function of x raised to a power of x, we *must* do this derivative using logarithmic differentiation.

$$\begin{aligned}
y &= (\sin x)^x \\
\ln y &= \ln[(\sin x)^x] \\
\ln y &= x\ln\sin x \\
\frac{d}{dx}[\ln y] &= \frac{d}{dx}[x\ln\sin x] \\
\frac{d}{dy}[\ln y]\cdot\frac{dy}{dx} &= \frac{d}{dx}[x]\ln\sin x + x\frac{d}{dx}[\ln\sin x] \\
\frac{1}{y}\;\frac{dy}{dx} &= (1)\ln\sin x + x\frac{\frac{d}{dx}[\sin x]}{\sin x} \\
\frac{dy}{dx} &= y\left(\ln\sin x + x\frac{\cos x}{\sin x}\right) \\
\frac{dy}{dx} &= (\sin x)^x\,(\ln\sin x + x\cot x)
\end{aligned}$$

Problem 11. We need to draw a little sketch of the situation, and set up our notation. I've drawn my sketch in *Mathematica*, but you should draw your sketch by hand.



Here, the man has moved a distance of x from the point P, and the woman has moved a distance y from Q. The distance between the man and woman will be z.

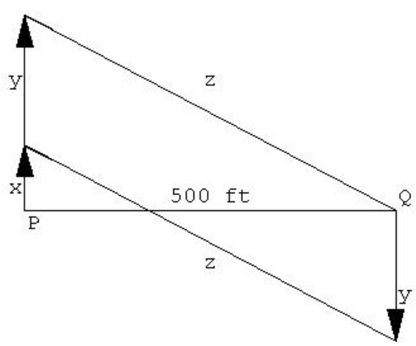
We are told some information, and we can convert that information into mathematical equations using our notation.

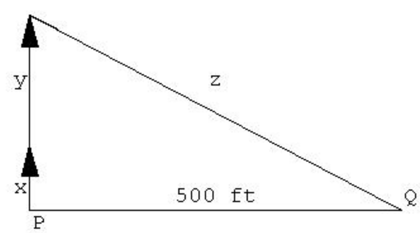
We are told that the man walks with speed $\left|\dfrac{dx}{dt}\right| = 4$ ft/s.

We are told that the woman walks with speed $\left|\dfrac{dy}{dt}\right| = 5$ ft/s.

What is unknown is the rate at which the are moving apart, which is the rate of change of the distance between them, which is $\left|\dfrac{dz}{dt}\right|$ in our notation.

To get the relation between x, y and z, I am going to redraw my diagram.





From the last diagram, we get the relation:

$$z^2 = (x + y)^2 + 500^2$$

What are x, y and z after the woman has been walking for 15 minutes?

$\dfrac{dy}{dt} = 5$ ft/s $= 300$ ft/min so in 15 minutes the woman has walked $y = 15 \cdot 300 = 4500$ ft.

$\dfrac{dx}{dt} = 4$ ft/s $= 240$ ft/min in 20 minutes the man has walked $x = 20 \cdot 240 = 4800$ ft.

The distance between them after the woman has been walking 15 minutes is $z = \sqrt{(x + y)^2 + 500^2} = \sqrt{(4800 + 4500)^2 + 500^2} = 100\sqrt{8674}$ ft.

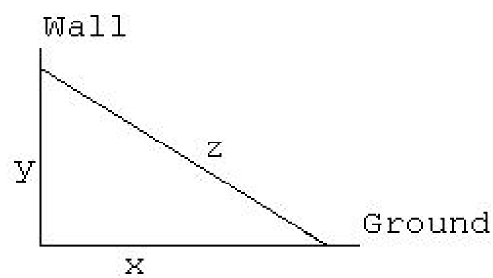
To introduce the rates of change, we must implicitly differentiate the relation with respect to time:

$$\begin{aligned}
z^2 &= (x+y)^2 + 500^2 \\
\frac{d}{dt}[z^2 &= (x+y)^2 + 500^2] \\
\frac{d}{dt}[z^2] &= \frac{d}{dt}[(x+y)^2] \\
2z\frac{dz}{dt} &= 2(x+y)\left(\frac{dx}{dt} + \frac{dy}{dt}\right) \\
\frac{dz}{dt} &= \frac{x+y}{z}\left(\frac{dx}{dt} + \frac{dy}{dt}\right)
\end{aligned}$$

The rate of change of distance between them after the woman has been walking for 15 minutes is

$$\frac{dz}{dt} = \frac{4500+4800}{100\sqrt{8674}}(300+240) = 25110\sqrt{\frac{2}{4337}} \text{ ft/min.}$$

Problem 12. We are given the rate of change of the distance the bottom of the ladder is from the wall. What is unknown is the rate of change of the distance of the top of the ladder from the ground. All our rates of change are with respect to time.
Diagram/Notation:



x is the distance of the base of the ladder from the wall.
y is the distance of the top of the ladder from the ground.
z is the length of the ladder. This is given as 12 ft.

$\frac{dx}{dt}$ is given as 2 ft/s.
$\frac{dy}{dt}$ is the unknown.
$\frac{dz}{dt}$ is zero since the ladder does not change length.
From the diagram, we get the relation

$$z^2 = x^2 + y^2$$

Implicitly differentiate with respect to time:

$$\begin{aligned}
\frac{d}{dt}[z^2 &= x^2 + y^2] \\
2z\,\frac{dz}{dt} &= 2x\frac{dx}{dt} + 2y\frac{dy}{dt} \\
\frac{dy}{dt} &= \frac{1}{y}\left(z\frac{dz}{dt} - x\frac{dx}{dt}\right)
\end{aligned}$$

Now we substitute in our values for when the bottom of the ladder is 6 ft from the wall:
$z = 12$ ft
$x = 6$ ft
$y = \sqrt{z^2 - x^2} = \sqrt{12^2 - 6^2} = 6\sqrt{3}$ ft
The rates of change dx/dt and dz/dt were found above.

$$\frac{dy}{dt} = \frac{1}{6\sqrt{3}}\left(12(0) - (6)(2)\right) = -\frac{2}{\sqrt{3}}$$

The top of the ladder is sliding down (that's what the minus sign means!) the wall at a rate of $2/\sqrt{3}$ ft/s when the bottom of the ladder is 6 ft from the wall.

Problem 13. This is done with implicit differentiation.

$$y = \arctan x \longrightarrow x = \tan y$$

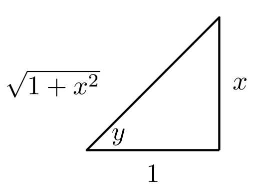
Now implicitly differentiate

$$\begin{aligned} x &= \tan y \\ \frac{d}{dx}[x &= \tan y] \\ \frac{d}{dx}[x] &= \frac{d}{dx}[\tan y] \\ 1 &= \frac{d}{dy}[\tan y] \cdot \frac{dy}{dx} \quad \text{chain rule} \\ 1 &= \sec^2 y \frac{dy}{dx} \\ \frac{dy}{dx} &= \frac{1}{\sec^2 y} = \cos^2 y \end{aligned}$$

Now we have to get this back into an explicit function of x, since that is what we were given.

Use SOH CAH TOA rules and a diagram of the situation to help us:

$$x = \tan y = \frac{x}{1} = \frac{\text{opposite}}{\text{adjacent}}$$



$$\cos y = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{1}{\sqrt{1+x^2}}, \quad \cos^2 y = \frac{1}{1+x^2}.$$

Therefore,

$$\frac{dy}{dx} = \frac{d}{dx}[\arctan x] = \frac{1}{1+x^2}$$