This is a set of practice test problems for Chapter 5. This is in **no way** an inclusive set of problems–there can be other types of problems on the actual test. The solutions are what I would accept on a test, but you may want to add more detail, and explain your steps with words–remember, I am interested in the process you use to solve problems!

There will be five problems on the test. Most will involve more than one part. You will have 100 minutes to complete the test. You may not use *Mathematica* or calculators on this test.

Useful Information that you will be given on this test:

$$R_n = \frac{b-a}{n}\sum_{i=1}^{n} f\Big(a + \frac{(b-a)}{n}\cdot i\Big), \quad L_n = \frac{b-a}{n}\sum_{i=1}^{n} f\Big(a + \frac{(b-a)}{n}\cdot (i-1)\Big)$$

1. Determine a region whose area is equal to the given limit. Do not evaluate the limit.

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{\pi}{4n}\tan\frac{i\pi}{4n}$$

2. Consider $\displaystyle\int_1^3 (x-1)\,dx$.

(a) Compute the integral by sketching the integrand and computing the area under the curve from the sketch.
(b) Compute the integral using the Midpoint Rule and two rectangles. Sketch the situation and explain why the Midpoint Rule gives the exact answer in this case.

3. Evaluate the integral $\displaystyle\int_{-1}^{0}\Big(-2-\sqrt{1-x^2}\Big)\,dx$ exactly by sketching the integrand and interpreting the integral in terms of areas.

4. Evaluate

(a) $\displaystyle\int t^2\cos(1-t^3)\,dt$ (b) $\displaystyle\int \frac{\cos(\ln x)}{x}\,dx$

5. Evaluate

(a) $\displaystyle\int (1-x)\sqrt{2x-x^2}\,dx$ (b) $\displaystyle\int \frac{e^{\sqrt{t}}}{\sqrt{t}}\,dt$

6. Evaluate

(a) $\displaystyle\int_0^{\pi/2}\frac{\cos x}{\sqrt{1+\sin x}}\,dx$ (b) $\displaystyle\int_2^3 \frac{x}{(x^2-1)^2}\,dx$

7. Evaluate

(a) $\displaystyle\int_0^1 xe^{-x^2}\,dx$ (b) $\displaystyle\int_0^{\pi/4}\frac{\sin\theta}{\cos^2\theta}\,d\theta$

8. An object moves along a line so that its velocity at time t (in m/s) is $v(t) = t^2 - 6t + 8$. Find the displacement and total distance traveled by the object for $0 \le t \le 8$. Displacement to the right is positive. You may leave your answer for the <u>total distance traveled</u> in the form: (Antiderivative)$\big|_a^b$.

9. A continuous, odd function $f(x)$ has the property $\displaystyle\int_{-a}^{a} f(x)\,dx = 0$.

(a) Sketch a graph to show geometrically why this is so.
(b) Prove this statement.

10. If $f(x)$ is continuous on $[0,1]$, prove that $\displaystyle\int_0^1 f(x)\,dx = \int_0^1 f(1-x)\,dx$.

Solutions

Problem 1. Compare

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{\pi}{4n}\tan\frac{i\pi}{4n}$$

to the definition of the integral using right hand endpoints in each subinterval:

$$\int_a^b f(x)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n}\frac{b-a}{n}f(a+\frac{b-a}{n}\cdot i)$$

We need to identify a, b, f. Let's rewrite the original expression, and see what we can pull out of it:

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{\pi/4}{n}\tan\frac{\pi/4}{n}\cdot i$$

From this, we might guess that

$$b-a=\frac{\pi}{4}$$

$$a=0$$

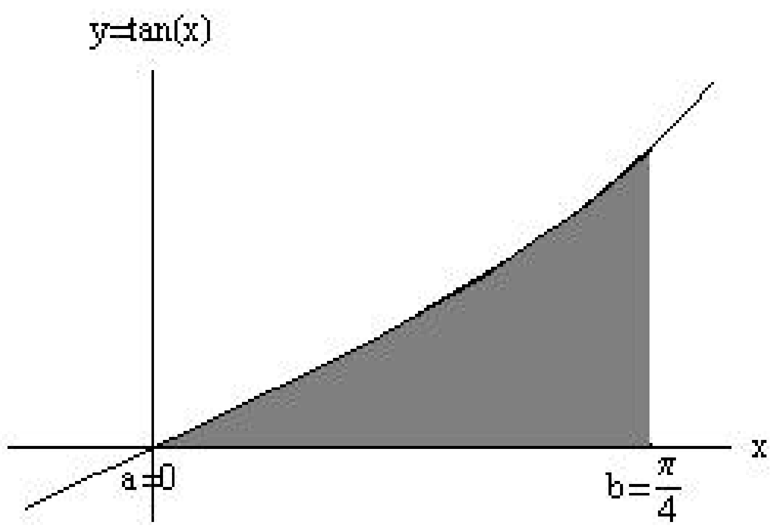
$$b=\frac{\pi}{4}$$

$$f(x)=\tan x$$

We can check if our guess is right by substituting:

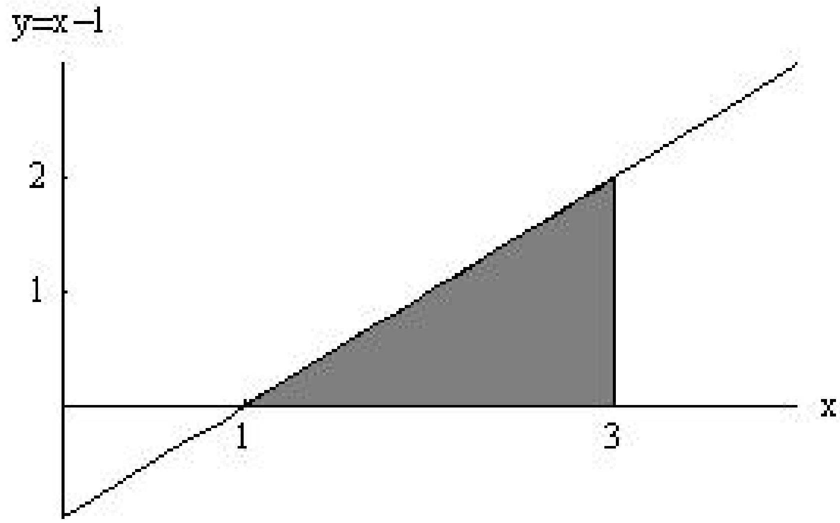
$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{\pi/4}{n}\tan(\frac{\pi/4}{n}\cdot i)$$

We have guessed correctly!
A sketch of the area this represents is given below:



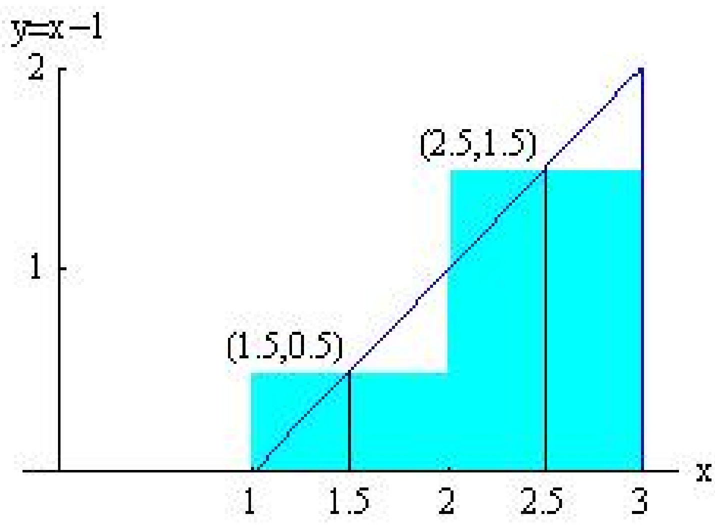
Problem 2. I have sketched the function $y = x - 1$ below. The integral $\int_1^3 (x-1)\,dx$ is represented by the area under the curve, the shaded area.

Since the curve is above the x-axis in the region $1 \le x \le 3$, we do not need to worry about subtracting areas to get the integral.



$$\int_1^3 (x-1)\,dx = \text{ shaded area } = \frac{1}{2}(\text{base})(\text{height}) = \frac{1}{2}(2)(2) = 2$$

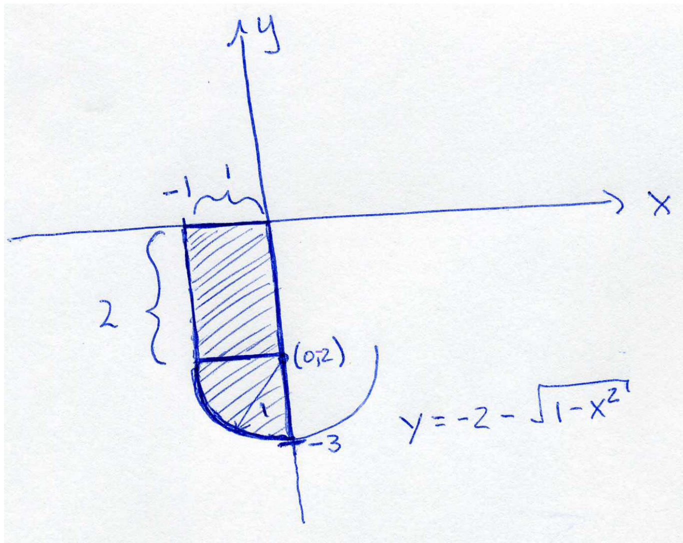
For part b, I have redrawn my sketch, this time including the two rectangles that make up M_2 (the point inside each subinterval is chosen to be the midpoint).



$$\int_1^3 (x-1)\,dx \sim \text{ area of rectangles } = (1)(0.5) + (1)(1.5) = 2.$$

This is the actual area since the area being missed is equal to the area being overestimated for each rectangle.

Problem 3. The integrand is $y = -2 - \sqrt{1-x^2}$, which simplifies to $(y+2)^2 + x^2 = 1^2$, which is a circle of radius 1 centered at $(0,-2)$. We only want the bottom half of the circle since we began with $-\sqrt{\ }$.



The integral is the shaded area, which is the area of the rectangle with width 1 and height 2 plus one quarter the area of a circle of radius 1. We also have to make this negative since it is below the x-axis.

$$\int_{-1}^{0} \left(-2 - \sqrt{1-x^2}\right) dx = -\left((1)(2) + \frac{1}{4}\pi(1)^2\right) = -2 - \frac{\pi}{4}.$$

Problem 4a.

$$\int t^2 \cos(1-t^3)\, dt \qquad \text{Substitute: } \begin{array}{l} u = 1-t^3 \\ du = -3t^2\, dt \end{array}$$

$$\begin{aligned}
\int t^2 \cos(1-t^3)\, dt &= \int \cos(1-t^3) t^2\, dt \\
&= \int \cos(1-t^3) \frac{(-3t^2 dt)}{(-3)} \\
&= -\frac{1}{3} \int \cos u\, du \\
&= -\frac{1}{3} \sin u + c \\
&= -\frac{1}{3} \sin(1-t^3) + c
\end{aligned}$$

Problem 4b.

$$\int \frac{\cos(\ln x)}{x}\, dx \qquad \text{Substitute: } \begin{array}{l} u = \ln x \\ du = \dfrac{dx}{x} \end{array}$$

$$\begin{aligned}
\int \frac{\cos(\ln x)}{x}\, dx &= \int \cos(\ln x) \frac{dx}{x} \\
&= \int \cos u\, du \\
&= \sin u + c \\
&= \sin(\ln x) + c
\end{aligned}$$

Problem 5a.

$$\int (1-x)\sqrt{2x-x^2}\,dx \qquad \text{Substitute: } \begin{array}{l} u = 2x - x^2 \\ du = (2-2x)\,dx \end{array}$$

$$\begin{aligned}
\int (1-x)\sqrt{2x-x^2}\,dx &= \frac{1}{2}\int \sqrt{2x-x^2}(2-2x)\,dx \\
&= \frac{1}{2}\int \sqrt{u}du \\
&= \frac{1}{2}\int u^{1/2}du \\
&= \frac{1}{2}\frac{u^{3/2}}{(3/2)} + c \\
&= \frac{1}{3}(2x-x^2)^{3/2} + c
\end{aligned}$$

Problem 5b.

$$\int \frac{1}{\sqrt{t}}e^{\sqrt{t}}\,dt \qquad \text{Substitute: } \begin{array}{l} u = \sqrt{t} \\ du = \frac{1}{2}t^{-1/2}\,dt = \frac{1}{2\sqrt{t}}\,dt \end{array}$$

$$\begin{aligned}
\int \frac{1}{\sqrt{t}}e^{\sqrt{t}}\,dt &= 2\int e^{\sqrt{t}}\,\frac{dt}{2\sqrt{t}} \\
&= 2\int e^u\,du \\
&= 2e^u + c \\
&= 2e^{\sqrt{t}} + c
\end{aligned}$$

Problem 6a.

$$\int_0^{\pi/2} \frac{\cos x}{\sqrt{1+\sin x}}\,dx \qquad \text{Substitute: } \begin{array}{l} u = 1 + \sin x \\ du = \cos x\,dx \\ \text{Change limits:} \\ \quad \text{when } x = 0, u = 1 \\ \quad \text{when } x = \frac{\pi}{2}, u = 2 \end{array}$$

$$\begin{aligned}
\int_0^{\pi/2} \frac{\cos x}{\sqrt{1+\sin x}}\,dx &= \int_0^{\pi/2} \frac{1}{\sqrt{1+\sin x}}\cos x\,dx \\
&= \int_1^2 \frac{1}{\sqrt{u}}\,du \\
&= \int_1^2 u^{-1/2}\,du \\
&= \left.\frac{u^{1/2}}{(1/2)}\right|_1^2 \\
&= 2(\sqrt{2}-1)
\end{aligned}$$

Problem 6b.

$$\int_2^3 \frac{x}{(x^2-1)^2}\,dx \qquad \text{Substitute: } \begin{array}{l} u = x^2 - 1 \\ du = 2x\,dx \\ \text{Change limits:} \\ \quad \text{when } x = 2, u = 3 \\ \quad \text{when } x = 3, u = 8 \end{array}$$

$$\begin{aligned}
\int_2^3 \frac{x}{(x^2-1)^2}\,dx &= \frac{1}{2}\int_2^3 \frac{1}{(x^2-1)^2}2x\,dx \\
&= \frac{1}{2}\int_3^8 \frac{1}{u^2}\,du \\
&= \frac{1}{2}\int_3^8 u^{-2}\,du \\
&= \left(\frac{1}{2}\right)\frac{u^{-1}}{(-1)}\bigg|_3^8 \\
&= -\frac{1}{2}\left[\frac{1}{u}\right]_3^8 \\
&= -\frac{1}{2}\left[\frac{1}{8}-\frac{1}{3}\right] \\
&= -\frac{1}{2}\left[\frac{3}{24}-\frac{8}{24}\right] \\
&= -\frac{1}{2}\left[-\frac{5}{24}\right] = \frac{5}{48}
\end{aligned}$$

Problem 7a.

$$\int_0^1 xe^{-x^2}\,dx \qquad \text{Substitute: } \begin{array}{l} u=-x^2 \\ du=-2x\,dx \\ \text{Change limits:} \\ \ \text{when } x=0, u=0 \\ \ \text{when } x=1, u=-1 \end{array}$$

$$\begin{aligned}
\int_0^1 xe^{-x^2}\,dx &= -\frac{1}{2}\int_0^1 e^{-x^2}(-2x)\,dx \\
&= -\frac{1}{2}\int_0^{-1} e^u\,du \\
&= -\frac{1}{2}\left[e^u\right]_0^{-1} \\
&= -\frac{1}{2}\left[e^{-1}-e^0\right] \\
&= -\frac{1}{2}\left[\frac{1}{e}-1\right]
\end{aligned}$$

Problem 7b.

$$\int_0^{\pi/4} \frac{\sin\theta}{\cos^2\theta}\,d\theta \qquad \text{Substitute: } \begin{array}{l} u=\cos\theta \\ du=-\sin\theta\,d\theta \\ \text{Change limits:} \\ \ \text{when } \theta=0, u=1 \\ \ \text{when } \theta=\pi/4, u=\cos\pi/4=\frac{1}{\sqrt{2}} \end{array}$$

$$\begin{aligned}
\int_0^{\pi/4} \frac{\sin\theta}{\cos^2\theta}\,d\theta &= -\int_0^{\pi/4}\frac{(-\sin\theta)d\theta}{\cos^2\theta} \\
&= -\int_1^{1/\sqrt{2}}\frac{du}{u^2} \\
&= -\int_1^{1/\sqrt{2}} u^{-2}du
\end{aligned}$$

$$
\begin{aligned}
&= -\left(\frac{u^{-1}}{(-1)}\right)_1^{1/\sqrt{2}} \\
&= \left(\frac{1}{u}\right)_1^{1/\sqrt{2}} \\
&= \frac{1}{1/\sqrt{2}} - 1 \\
&= \sqrt{2} - 1
\end{aligned}
$$

Note: If you forget that $\cos \pi/4 = 1/\sqrt{2}$, draw a little unit circle and work out the value using geometry.

Problem 8.

$$
\begin{aligned}
\text{displacement} &= s(8) - s(0) \\
&= \int_0^8 v(t)\, dt \\
&= \int_0^8 (t^2 - 6t + 8)\, dt \\
&= \left(\frac{t^3}{3} - 3t^2 + 8t\right)_0^8 \\
&= \left(\frac{8^3}{3} - 3(8)^2 + 8(8)\right) - (0 - 0 + 0) \\
&= \frac{512}{3} - 192 + 64 = \frac{128}{3} \text{ m}
\end{aligned}
$$

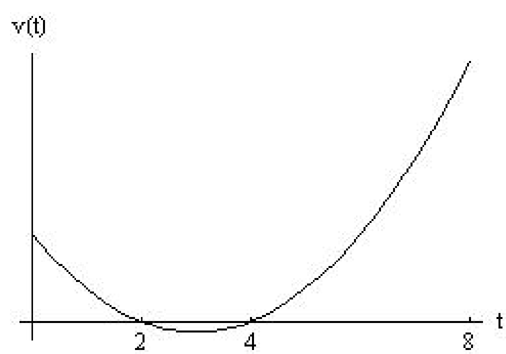
The particle moved 128/3 m to the right (because the answer was positive).

$$
\text{distance traveled} = \int_0^8 |v(t)|\, dt
$$

We need to work out the absolute value. In this case (quadratic), it is easiest to do by finding the roots of the quadratic. If you can't find the factorization by inspection, you can use the quadratic formula.

$$
v(t) = t^2 - 6t + 8 = (t - 4)(t - 2)
$$

Since $v(t)$ is a parabola opening up, we know that it must look something like the following:

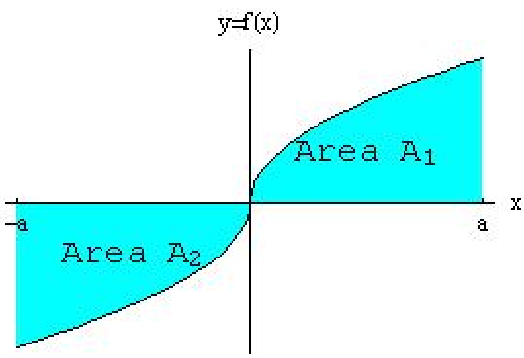


From this graph, we can figure out when the function is positive, and when it is negative.

This allows us to rewrite the integral as follows:

$$
\begin{aligned}
\text{distance traveled} &= \int_0^8 |v(t)|\,dt \\
&= \int_0^8 |t^2-6t+8|\,dt \\
&= \int_0^2 |t^2-6t+8|\,dt + \int_2^4 |t^2-6t+8|\,dt + \int_4^8 |t^2-6t+8|\,dt \\
&= \int_0^2 (t^2-6t+8)\,dt - \int_2^4 (t^2-6t+8)\,dt + \int_4^8 (t^2-6t+8)\,dt \\
&= \left(\frac{t^3}{3}-3t^2+8t\right)_0^2 - \left(\frac{t^3}{3}-3t^2+8t\right)_2^4 + \left(\frac{t^3}{3}-3t^2+8t\right)_4^8
\end{aligned}
$$

Problem 9. Since $f(x)$ is odd, we know that $f(-x) = -f(x)$. This means the function must be symmetric with respect to rotation of 180 degrees about the origin. Graphically, this means the function looks something like:



The shaded area A_1 is equal to the shaded area A_2. However, A_1 is below the x axis. Therefore, the two areas cancel in the integration, and

$$\int_{-a}^{a} f(x)\,dx = 0.$$

To prove this result, we need to work with the integral and do a substitution:
If f is odd, then $f(-x) = -f(x)$.

$$
\begin{aligned}
\int_{-a}^{a} f(x)\,dx &= \int_{-a}^{0} f(x)\,dx + \int_0^a f(x)\,dx \\
&= -\int_0^{-a} f(x)\,dx + \int_0^a f(x)\,dx \quad \text{First Integral Substitution: } \begin{array}{l} u=-x \\ du=-dx \\ \text{Change Limits:} \\ \ \text{when } x=0 \to u=0 \\ \ \text{when } x=-a \to u=a \end{array} \\
&= \int_0^a f(-u)\,du + \int_0^a f(x)\,dx \\
&= -\int_0^a f(u)\,du + \int_0^a f(x)\,dx \text{ (since } f \text{ is odd)} \\
&= -\int_0^a f(x)\,dx + \int_0^a f(x)\,dx \text{ (Substitution: } u=x \text{ (nothing will change))} \\
&= 0
\end{aligned}
$$

Problem 10.

$$\int_0^1 f(x)\,dx \qquad \text{Substitute: } \begin{array}{l} u = 1 - x \longrightarrow x = 1 - u \\ du = -dx \longrightarrow dx = -du \\ \text{Change limits:} \\ \quad \text{when } x = 0, u = 1 \\ \quad \text{when } x = 1, u = 0 \end{array}$$

$$\begin{aligned} \int_0^1 f(x)\,dx &= -\int_0^1 f(x)\,(-dx) \\ &= -\int_1^0 f(1-u)\,du \\ &= \int_0^1 f(1-u)\,du \quad (\text{Substitution: } u = x \text{ (nothing will change)}) \\ &= \int_0^1 f(1-x)\,dx \end{aligned}$$